STUDIES ON THE STRUCTURE OF INPUT-OUTPUT MODELS FOR NATIONAL, REGIONAL, AND MULTIREGIONAL ECONOMIC ANALYSIS

by

KIRKOR BOZDOGAN

B.S., University of Wisconsin
(1961)

M.C.P., Harvard University
(1963)

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Signature of Author ______________________________
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Certified by ______________________________
Thesis Supervisor

Accepted by ______________________________
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ABSTRACT

Title of the Thesis: Studies on the Structure of Input-Output Models for National, Regional, and Multiregional Economic Analysis

Name of the Author: Kirkor Bozdogan

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The input-output approach in economics, which is a theoretical and empirical extension of the classical theory of interdependence, is central today to many aspects of quantitative economic research, particularly to economic impact analysis, multisectoral economic forecasting, and programming for economic development. The purpose of this thesis, broadly defined, is to study the mathematical and empirical structure of input-output models as they have been developed and used in the past, to develop two alternative model formulations that represent substantial improvements over the existing models, to study the empirical properties of these proposed model formulations by conducting a series of experiments, and finally, to explore systematically the mathematical structure of further extensions of the proposed models for national, regional, and multiregional economic analysis, under alternative assumptions on information availability, the measurement of the *technology* matrices, and the treatment of competitive and noncompetitive imports.

In the empirical part of the thesis, two alternative models, designated as the *commodity technology* and *industry technology* models, are mathematically developed, first under the assumption that competitive imports are treated endogenously and then under the assumption that they are treated exogenously. These two models, which are formulated in response to the inadequate empirical structure of the models developed in the past, are estimated for 1958 for the United States economy, using information made available by the National Planning Association and the U.S. Office of Business Economics. Using both models (with competitive imports treated exogenously), predictions of intermediate demand for domestically produced products and domestic sectoral *product* output levels are obtained for 1961. This process is replicated at four different aggregation levels for both models (i.e., 79x79, 60x60, 45x45, and 17x17). The results are used to investigate the comparative

predictive performance of the two models, the nature of the *aggregation bias,* and the detailed structure of prediction errors.

The results of these experiments can be summarized as follows. *First,* the *industry technology* model gives a generally better performance, over all aggregations taken together, than the *commodity technology* model, which should lead to a skeptical view of the traditional input-output theory. Such a conclusion, however, must be carefully qualified, since the secondary product adjustment procedure used in developing the basic information underlying the *commodity technology* model appears to have caused this model's somewhat inferior performance. It seems certain, at the same time, that so long as the basic information required in input-output model construction is available not in terms of *product-to-product* flows but in terms of *product-to-industry* flows, the *industry technology* model serves just as well and perhaps even better. *Secondly,* the superiority of either model at a given level of aggregation is difficult to judge, since they both display strengths, as well as weaknesses, at different aggregation levels. The choice of one model over the other at a given level of aggregation, therefore, pretty much depends on the researcher's own preferences as to what type of prediction error he would like to see minimized. Further, the respective qualities of the two models cannot be assessed in an unequivocal way, particularly in the presence of measurement errors inherent in the two models that remain unknown. *Thirdly,* there appears to be a clearcut nonlinear functional relationship, approximating a reverse-J shape, between the aggregation levels at which the models are built and the respective measures of overall predictive performance. *Finally,* the absolute values of input-output prediction errors show a significant degree of correlation with the amounts that are actually to be predicted.

Throughout the thesis, an attempt has been made to keep it as self-contained as possible. To this end, the five chapters that make up the main text of the thesis are accompanied by six appendices containing unified discussions of topics in input-output analysis that presently are scattered in numerous periodicals in as many languages.

Thesis Supervisor: Jerome Rothenberg

Title: Professor of Economics

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Without the considerable amount of support I have received from Mr. Harry G. Foden, Dr. A. George Gols, and Mr. John Magee at Arthur D. Little, Inc. (ADL), it would have been very difficult for me financially or otherwise to undertake the research reported in this thesis. I am particularly grateful to Mr. Harry G. Foden for his understanding and help over the years.

Much of the data used in the input-output experiments reported here came from the U.S. Office of Business Economics, in the form of magnetic tapes and preliminary tabulations. I should like to thank Mr. Martin Marimont of the National Economics Division, U.S. Office of Business Economics, for his help and cooperation, as well as for his active interest in my research. My thanks and appreciation also extend to Mr. Marshall K. Wood of the National Planning Association, who, through the written approval of Dr. Joseph D. Coker of the U.S. Office of Emergency Planning, made available the data used in generating what I have called the *commodity technology* matrices.

The computer operations that were required in the input-output experiments proved to be quite complicated and prolonged. Without the computer programming help I received from Mr. John Forlizzi, Mr. Miguel Suarez, Mrs. Mimi Wong, and Mr. John Getsinger at different phases of the process, it would have been virtually impossible for me to conduct the experiments reported in Chapter IV. I would especially like to thank Mr. John Getsinger, who re-wrote many parts of the program, tested it, and made it fully operational under severe time constraints.

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Mrs. Glenda Washburn, my secretary at ADL, typed earlier rough drafts and performed other tasks with exceptional dedication and skill. Mrs. Violet Gaumont was responsible for the typing of the final draft for printing. I value and appreciate the superlative typing job done by her and by her colleagues, and want to thank Mr. Richard J. Callahan for his help in proofreading and in many other respects. My special thanks go to Mr. Joshua Tolford for his help with graphics, and to Messrs. William J. Gallagher and Joseph Sammon for their help and cooperation in having the thesis printed at the ADL report processing center.

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None of the persons mentioned above should be held responsible for the shortcomings of this work. In the final analysis, I cheerfully accept all blame for the inadequacies and errors that may still remain.

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CHAPTER I

INTRODUCTION: THE BASIC MODEL IN ITS ENVIRONMENT

A. INTRODUCTION : AN OVERVIEW

The input-output approach in economic analysis is a theoretical and empirical extension of the classical theory of interdependence. Today, the idea of general interdependence among many parts of an economic system is central to modern economic analysis and mathematical programming for economic development. In recent years, the emergence of *mixed* models, using the input-output system as an integral part of a macroeconomic model built for short or long-term economic forecasting or for use as a quantitative policy model, marks an important new epoch in economic analysis. The literature on input-output analysis, rather than slowing down, has been increasing in recent years at such an impressive rate that periodic bibliographical references are required just to keep pace with new developments in the field. With the increasing demand now put on input-output models for economic forecasting, programming for economic development, or for policy formulation, this is an appropriate time to take stock of the empirical structure of input-output models and to introduce some long overdue reforms in the empirical construction and mathematical formulation of input-output models. In a general sense, this dissertation does just that.

Previous experimental studies in the input-output field have been mainly preoccupied with the temporal constancy of the *structural* parameters of input-output models and/or with the predictive performance of input-output models vis-à-vis a series of simpler, *naive* methods. With rare exceptions, these studies suffer from a number of serious shortcomings as scientifically valid experiments, as documented in this dissertation. It should not come too much as a surprise to say that their cumulative scholarly contribution has been quite modest.

Today, the temporal constancy of the *technological* coefficients is no longer the dominant issue in input-output analysis. Methods now exist by which to keep the existing input-output models reasonably up-to-date. Further, with the emergence of *mixed* models, industry specialists have come to play an increasingly significant role in input-output research, with the consequence that it is now generally possible to monitor and anticipate the changes in technological and other factors that cause shifts in the model's parameters.

More important now than ever before is a re-examination of the theoretical and empirical structure of input-output models and a reformulation of the system in alternative ways in response to a whole set of empirical and other circumstances that, in the past, have been recognized and dealt with either too casually or not at all. In this dissertation, it is argued that the *traditional* structure of input-output models, particularly those of the United States economy developed and published in the past, contain what amounts to deliberate errors in the measurement of the structural parameters (i.e., the *technology* matrix) because of a preoccupation not so much with accuracy in the measurement of the structural parameters but with achieving sectoral commodity balances in the models, and that, consequently, the published results have reduced utility for economic analysis. It is then argued that since the basic information used in developing input–output models is collected in most countries in terms of *produce–to–industry flows* of goods and services, with each industry defined as a set of *establishments* producing both their primary and secondary products, two realistic alternatives are available in input-output model construction. First, the model can be developed under the assumption of a *commodity technology,* in which case certain methods, as suggested, can be used to transform the existing *product-to-industry flows* system into the theoretically more desirable *product-to-product flows* system. Secondly, the model can be developed under the assumption of an *industry technology*, in which case the existing *product-to-industry flows* information can be readily used, provided that separate *make* and *mix* matrices be incorporated into the mathematical formulation of the model. While the *make* matrix, in which rows indicate industries and columns represent products, shows the amount (value) of each produce made by different industries, the *mix* matrix shows the proportion of each product made by different industries. It is shown, further, that the assumptions made with respect to the treatment of competitive imports lead to alternative model formulations based on either *commodity technology* or *industry technology* assumptions. These alternative model formulations are then mathematically derived, empirically estimated, and used in a series of experiments.

The experiments reported in this dissertation are a series of sensitivity experiments to ascertain how input-output prediction errors are affected by alternative specifications of the model's structural parameters. Secondly, the sensitivity experiments are replicated at different levels of aggregation to find out the extent to which input-output prediction errors that are directly attributable to aggregation show systematic variations that convey an understanding of the *information loss* due to aggregation. In addition, an effort is made to investigate whether or not input-output prediction errors vary systematically with a set of variables that describe certain fundamental empirical characteristics of the model.

The input-output models developed and used in these experiments refer to the United States economy as a whole. The base year for which these models have been constructed refer to 1958, for which an input-output study of the United States has been conducted by the U. S. Office of Business Economics with the help of other government agencies. The input-output predictions obtained in these experiments are for 1961, the only year following 1958 for which the type of information required in these experiments was available at the time this study commenced. The basic data used in developing the models have been made available by the U. S. Office of Business Economics and the National Planning Association, as acknowledged in the Preface.

Following these sensitivity experiments, the models used in them are set forth, in a systematic manner, within the context of other models that can be developed and used for national and regional economic analysis, under alternative assumptions regarding information availability, the estimation of the technology matrices, and the treatment of competitive imports. Following this, an exploration is made of the structure of input-output models for multiregional economic analysis, by demonstrating, through a specific example, how the model formulations given earlier can be extended for multiregional analysis.

The organization of the dissertation is as follows. *First* this chapter sets forth the focus of the dissertation, following a brief historical note and a concise statement of some of the fundamental properties of the basic model, and provides a mathematical exposition of the effects of measurement errors on input-output predictions, as well as an exposition of the aggregation problem in input-output analysis. The *second* chapter argues, in effect, for the adoption of new methods and procedures in the empirical construction of input-output models, after presenting a critical assessment of some important conceptual and empirical problems faced in input-output model construction and the methods used to overcome them. The alternative approaches that are suggested pertain to the *commodity technology* and *industry technology* options that can be followed and the consequent model formulations to which they respectively lead. The *third* chapter sets forth the focus of the experiments reported in this dissertation within the context of past experiments on input-output models, after examining the major findings of these past experiments and documenting their shortcomings. The *fourth* chapter, then, contains an explanation of the design and analysis of the sensitivity experiments the focus of which has already been described and which comprise the major empirical work reported in this dissertation. Finally, in the *fifth* or last chapter, a systematic exploration is made of alternative model formulations for national, regional, and multiregional economic analysis, under alternative assumptions regarding information availability, the measurement of the *technological* coefficients matrices, and the treatment of competitive imports.

An effort is made in this dissertation to make it as self-contained as possible. This is mostly achieved by providing extended discussions in the appendices, which serve not only to supplement the respective chapters but also contain expositions that are difficult to find elsewhere in the literature. Appendix A, for example, contains a unified discussion of the mathematical properties of the basic open model that represents perhaps the most exhaustive and unified exposition of the subject to date. Drawing heavily upon the mathematical literature, the discussion concentrates on the properties of positive and non-negative matrices and M–matrices, on the one hand, and on the relationships between such matrices and the Minkowski–Leontief (i.e., *technological* coefficients), Leontief, and Leontief inverse matrices, on the other. Further, the appendix contains an exposition of alternative necessary and sufficient conditions for the existence of a nonnegative Leontief inverse, including the Hawkins–Simon condition, and such additional topics as the decomposability and triangulation of input-output matrices, the convergence of the infinite multiplier chains to the Leontief inverse, and the economic meaning of this convergence process.

Similarly, Appendix B contains a discussion of a set of related topics in input-output analysis that can be found in the literature only in fragments. These topics include (a) the inverse of the Leontief inverse and the *matrix multipliers,* which is a more extensive discussion than given in Appendix A, (b) the relationship between final demand, sectoral output requirements, and value added *income generation*, (c) the substitution theorem, (d) the Leontief Paradox, (e) prices in the open input-output system, (f) the closed input-output model, (g) the dynamic input-output model, and (h) the relationship of the basic open model to linear programming. The content of the other appendices are described in the respective chapters to which they correspond.

B. A BRIEF REVIEW OF HISTORICAL BACKGROUND

The notion of the interdependence of economic activities was first originated by Francois Quesnay, who presented his famous *Tableau Economique* in 1758.[1] Later, Leon Walras elaborated upon Quesnay's work in developing his system of general economic equilibrium.[2] The idea of general interdependence among many parts of an economic system has become by now the very foundation of modern economic analysis and mathematical programming for regional and national economic development.

Input-output analysis, as developed by Wassily Leontief, is a theoretical and empirical extension of the classical theory of general interdependence, which encompasses the entire economy of a region, country, or groups of countries, as a single economic system and attempts to depict quantitatively all of its activities in terms of the specific measurable properties of its structure.[3] The *structural* approach, represented by input-output analysis, is neither purely descriptive nor even strictly behavioristic, but in a general sense analytical. As a powerful analytical tool, the input-output method yields a comprehensive understanding of the economic system in a process of simultaneous adjustment, in which the interdependent flows of production, distribution, consumption, and investment are constantly affecting each other through a highly complex nexus of interlocking relationships.[4]

Leontief started to do empirical input-output research on the American economy in 1931 and published his first results in 1936[5] and 1941.[6] He characterized his research as "an attempt to construct, on the basis of available statistical materials, a *Tableau Economique*

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[1]Francois Quesnay, *Tableau Economique* (reproduction; London: British Economic Association, 1894). Also see Almerin Phillips, "*The Tableau Economique* as a Simple Leontief Model," *The Quarterly Journal of Economics,* Vol. 69, No. 1 (February, 1955), 137-144.

[2]Leon Walras, *Elements of Pure Economics,* Trans. William Jaffe (Homewood, Ill.: Richard D. Irwin, Inc., 1954).

[3]Wassily Leontief, *Input-Output Economics* (New York: Oxford University Press, 1966), p. vii.

[4]Wassily Leontief *et al., Studies in the Structure of the American Economy* (New York: Oxford University Press, 1953), p. v.

[5]Wassily Leontief, "Quantitative Input-Output Relations in the Economic System of the United States," *The Review of Economics and Statistics,* XVIII, 3 (August, 1936), 105-125.

[6]Wassily Leontief, *The Structure of the American Economy,* 1919-1939 (New York: Oxford University Press, Second Edition, 1951). The first three parts of this book reproduce the text of the 1951 edition without change. Part IV comprises four chapters which originally appeared in articles on the application of the input-output system.

of the United States for 1919 and 1929."[7] This effort culminated in the preparation of two 41-order (sector) input-output tables for 1919 and 1929. Between 1942 and 1944, another and this time a larger table (96-sector) was prepared for 1939 by the U. S. Bureau of Labor Statistics (BLS) under the direction of Leontief.[8]

The applicability of this first generation of input-output models to such problems as economic forecasting and economic development programming was hindered by the lack of electronic computers. During the period 1936 to 1940, for example, the best that one could hope for was to solve a set of simultaneous equations with twenty variables and twenty unknowns.[9] Since the solution of a large number of simultaneous equations normally requires matrix inversion,[10] this meant that the inverse of these earlier tables could not be computed efficiently unless the sizes of the matrices were considerably scaled down through aggregation. The 1939 table, for example, was later aggregated into a 42-sector table, a typical general solution of which took 56 hours on the Harvard Mark II computer.[11]

Empirical work in interindustry relations or input-output analysis developed rapidly during the postwar period, both in the United States and abroad. The Interindustry Relations Study of 1947 in the United States, undertaken by the U. S. Bureau of Labor Statistics and the Interagency Committee on Input-Output, marked perhaps the most massive and elaborate work performed anywhere in the field of input-output analysis.[12] The 450-order 1947 input-output table was not published. Instead, it was aggregated into the 200-order *Emergency Model,*[13] which was used to a limited extent during the Korean War in evaluating mobilization planning problems, and was followed by the partially

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[7] *Ibid.,* p. 9.

[8] W. Duane Evans and Marvin Hoffenberg, "The Interindustry Relations Study for 1947," *The Review of Economics and Statistics,* XXXIV, 2 (May, 1952), 97–142.

[9] George B. Dantzig, *Linear Programming and Extensions* (Princeton, New Jersey: Princeton University Press, 1963), p. 17.

[10] Some of the methods not requiring matrix inversion, such as the Seidel method and the familiar power series expansion method, will be discussed later.

[11] Leontief, *op. cit.* p. 26. [*Input-Output Economics*]

[12] For a detailed discussion of this study, see Evans and Hoffenberg, *op. cit.*

[13] U. S. Department of Labor, Bureau of Labor Statistics, *Table I – Interindustry Flow of Goods and Services by Industry of Origin and Destination: Continental United States, 1947* (October , 1952).

updated and slightly more disaggregated *Mobilization Model,* which was never made operational. The most well known product of the 1947 study became the more aggregated 50-order input-output table that was developed for government and private use.[14]

After such auspicious beginnings, large-scale empirical input-output research in the United States came to a virtual halt. After about a decade of inaction, and perhaps disinterest, the U. S. Government finally decided to prepare another table for 1958. Undertaken largely by the U. S. Office of Business Economics (OBE),[15] under the government-wide Interagency Growth Study Project, the results of this study have been made available only very recently, in a series of articles appearing in the *Survey of Current Business.*[16] For the first time, in the 1958 Input-Output Study,[17] U. S. national income accounts were fully

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[14] For this table, refer to Evans and Hoffenberg, *op. cit.*

[15] Other agencies of the federal government contributing to this study were (a) U. S. Department of Agriculture, Economic Research Service, Farm Income Branch, and (b) U. S. Department of the Interior, Bureau of Mines, Division of Economic Analysis.

[16] See, for example, the following:

Morris R. Goldman, Martin L. Marimont, and Beatrice N. Vaccara, "The Interindustry Structure of the United States, A Report on the 1958 Input-Output Study," *Survey of Current Business,* XLIV, 11 (November, 1964), 10–20;

Norman Frumkin, "Construction Activity in the 1958 Input-Output Study," *Survey of Current Business,* XLV, 5 (May, 1965), 13–24;

National Economics Division Staff, "The Transactions Table of the 1958 Input-Output Study and Revised Direct and Total Requirements Data," *Survey of Current Business,* XLV, 9 (September, 1965), 33–49, 56;

Nancy W. Simon, "Personal Consumption Expenditures in the 1958 Input-Output Study," *Survey of Current Business,* XLV, 10 (October, 1965), 7–14;

National Economics Division Staff, "Additional Industry Detail for the 1958 Input-Output Study," *Survey of Current Business,* XLVI, 4 (April, 1966), 14–17.

[17] This study is also known as "The 1958 Interindustry Structural Relations Study of the United States Economy" or as "The 1958 Interindustry Sales and Purchase (ISP) Study."

integrated with input-output accounts. Probably as a result of certain discrepancies observed in integrating the two systems, U. S. national income accounts have been revised all the way back to 1929.[18]

The 1958 Input-Output Study spearheaded a renewed interest and activity in empirical interindustry research. The Division of Economic Analysis of the U. S. Bureau of Mines, for example, disaggregated the six mining sectors appearing in the 1958 table into 46 mining activities, and recently computed the inverse of a 124-order table, in which the remaining industrial sectors are identical with those of the original 1958 table.[19] The 1958 table has recently been updated to 1961 by the OBE.[20] A much more ambitious input-output study for 1963 by the OBE has been underway for the last few years and its preliminary results are expected to be published fairly soon. Meanwhile, a major effort has been made by the Interagency Growth Study Project, in collaboration with other government agencies and private research organizations, to use the input-output framework for developing projections of the U. S. economy in considerable industry detail under alternative assumptions regarding rates and patterns of growth.[21]

Generally speaking, input-output modeling has met with more widespread acceptance and support abroad than in the United States, first in the European countries following World War II and then in the developing countries. A comprehensive survey of these earlier applications in many countries can be found in a standard text by Chenery and Clark.[22] Today, almost every developing country either already has an input-output table

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[18]The revised and benchmark national income and product estimates consistent with the 1958 input-output table are described in an article by the staff of the Office of Business Economics (OBE), "The National Income and Product Accounts of the United States, Revised Estimates, 1929–1964," *Survey of Current Business,* XLV, 8 (August 1965), 6-56.

[19]Kung-Lee Wang and Robert G. Kokat, *The Interindustry Structure of the U.S. Mining Industries, 1958,* U.S. Department of the Interior, Bureau of Mines, Information Circular 8338 (Washington, D.C.: U.S. Government Printing Office, 1967).

[20]U.S. Department of Commerce, Office of Business Economics, *Input-Output Transactions: 1961,* Staff Working Paper in Economics and Statistics, No. 16 (July 1968).

[21]U.S. Department of Labor, Bureau of Labor Statistics, *Projections 1970, Interindustry Relationships, Potential Demand, Employment,* BLS Bulletin No. 1536 (Washington, D.C.: U.S. Government Printing Office, 1966).

[22]Paul G. Clark and Hollis B. Chenery, *Interindustry Economics* New York: John Wiley and Sons, Inc., Fourth Printing, February 1965), 183-200.

or it is in the process of preparing one.[23] It is by now well established, as a matter of practical and theoretical necessity, that the task of constructing and updating input-output tables comprises the first important step in the process of programming for economic development.

The application of mathematical programming techniques to the problem of economic development is a fairly recent phenomenon. A review of the constantly increasing literature in this area indicates that linear programming has gained wide-scale acceptance as the main technique for studying and determining alternative allocations of investment, by explicitly taking into account the interdependence of investment decisions.[24]

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[23] For some relatively recent examples and general discussions, see the following:

Manuel Balboa, "Construction and Use of Input-Output Tables in Latin American Countries," in Tibor Barna (ed.), *Structural Interdependence and Economic Development,* Proceedings of an International Conference on Input-Output Techniques, Geneva, September 1961 (London: MacMillan and Co., Ltd. and New York: St. Martin's Press, 1963), 145–262;

Michael Bruno, "Some Applications of Input-Output Techniques to the Analysis of the Structure and Development of Israel's Economy," *ibid.,* 224–241;

Gamal E. Eleish, "The Input–Output Model in a Developing Economy: Egypt," *ibid.,* 199–220;

John C. H. Fei, "A Preliminary Input-Output Table for Large-Scale Industries in Pakistan," *The Pakistan Development Review,* II, 1 (Spring, 1962), 47–83;

A. Kundu, Interindustry Table for the Economy of British Guiana, 1959 and National Accounts, 1957–1960, Supplement to XII, 1, *Social and Economic Studies* (March, 1963).

[24] For a very small, practically random, sample see the following:

Michael Bruno, "Optimal Patterns of Trade and Development," *The Review of Economics and Statistics,* XLIX, 4 (1967), 545–554;

Hollis B. Chenery, *The Use of Interindustry Analysis in Development Programming,* in Tibor Barna (ed.), *op. cit.,* pp. 11–27;

Richard S. Eckaus and K. S. Parikh, *Planning for Growth: Multi-sectoral, Intertemporal Models Applied to India* (Cambridge, Mass.: Massachusetts Institute of Technology, Center for International Studies, Mimeographed, 1966);

A. S. Manne, "Key Sectors of the Mexican Economy, 1960–1970," in A. S. Manne and H. Markowitz, *Studies in Process Analysis* (New York: McGraw-Hill Book Co., Inc., 1963);

Jeffery B. Nugent, *Programming the Optimal Development of the Greek Economy, 1954–1961,* Center of Planning and Economic Research, Research Monograph Series, No. 15 (Athens: Constantinidis and Mihalas, 1966);

J. Sandee, *A Long-Term Planning Model for India* (New York: United Nations, 1959).

The input-output framework has provided the impetus for a multitude of theoretical and applied studies in economic interdependence. Rather than slowing down, the input-output literature has been growing in recent years at such an impressive rate that periodic bibliographies are required just to keep abreast of new developments in the field.[25] Only a few years ago, it used to be *the thing to do* to draw distinctions between input-output analysis and *econometric methods.* Such a distinction is as obsolete now as it might have been natural then. In this context, the analogy recently drawn by Almon between the input-output model and the saxophone is quite relevant:

> When Antoine Sax developed the saxophone around 1840, it was at first used only as a solo instrument: *saxophone* music could be clearly distinguished from other music. Today, of course, the saxophone is an integral part of any band. Until a few years ago, input-output was a sort of solo instrument played only by a few accomplished artists and with no more than an accompaniment from the rest of economic analysis. Today, it is more widely understood and accepted; but though it often sounds the theme, it now harmonizes with all the other instruments.[26]

The emergence in recent years of *mixed* models, using the input-output system as an integral part of a macroeconomic model built for short or long-term economic forecasting or for use as a quantitative economic policy model, marks undoubtedly a most important development in economic science. A complete list of such *mixed* models would be quite formidable, since "they are popping out of the ground everywhere at the same time, like the crocuses in Cambridge."[27] Numerous attempts at *mixed* model construction have

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[25] See, for example, the following:

Vera Riley and R. L. Allen, *Interindustry Economic Studies: A Comprehensive Bibliography on Interindustry Research* (Baltimore: Johns Hopkins University Press, 1955);

Charlotte E. Taskier, *Input-Output Bibliography, 1955–1960* (New York: United Nations, 1961);

United Nations, Statistical Office, *Input-Output Bibliography,* 1960–1963 (New York: United Nations, 1964);

____________, *Input-Output Bibliography, 1963-1966* (New York: United Nations, 1967).

[26] Clopper Almon, Jr., "New Developments in Input-Output Forecasting," paper presented at meetings of the Institute for Management Science in Boston (April 7, 1967), p. 3.

[27] *Ibid.,* p. 1. The reference is to Cambridge, Mass., although Cambridge, England would probably also qualify with its daffodils.

been made abroad.[28] In this country, Almon's own work is illustrative of these *mixed* forecasting models.[29] The work of the Interagency Growth Study Project in this country, mentioned earlier, represents another example.[30] The Brookings-SSRC econometric model of the United States,[31] which is an elaborate operational policy model with short-run stabilization focus, is so designed that it is susceptible of further disaggregation by sectors of the economy and/or by regions and local areas.[32] Lastly, the Arthur D. Little, Inc. (ADL) long-term economic forecasting model, jointly supported by a sizeable group of large American corporations, combines some of the best features of current econometric models with a comprehensive and systematic attempt at forecasting technological changes affecting input-output relationships.

In the context of these recent developments, this thesis focuses on a number of major problems pertaining to the empirical construction and application of input-output models that still remain by-and-large unresolved, through a series of experiments. These

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[28] A well known example is provided by the Cambridge model of economic growth, which has been described in a series of publications by the Cambridge University Department of Applied Economics. For an excellent discussion of this work, plus a fairly complete listing of relevant earlier publications, see Richard Stone, "The Analysis of Economic Systems", in Pontificiae Academiae Scientiarvm Scripta Varia, *Study Week on the Econometric Approach to Development Planning,* Results of a Conference held at the Vatican, October 7–13, 1963 (Chicago: Rand McNally and Co. and Amsterdam: North-Holland Publishing Co., 1965), pp. 3–88. For an earlier discussion of the Cambridge model, see Richard Stone and J. A. C. Brown, "A Long-Term Growth Model for the British Economy," in R. C. Geary (ed.), *Europe's Future in Figures* Amsterdam: North-Holland Publishing Co., 1962), pp. 287–310.

For another example of *mixed* models, see J. Waelbroeck, "Meccano: or a do-it-yourself approach to long-term forecasting *[sic],*" *Cahiers Economiques de Bruxelles,* (1966), 203–242.

[29] Clopper Almon, Jr., *The American Economy to 1975* (New York: Harper and Row, Publishers, 1966).

[30] U. S. Bureau of Labor Statistics, *op. cit.* *[Projections 1970...].*

[31] J. S. Duesenberry, G. Fromm, L. R. Klein, and E. Kuh (eds.), *The Brookings-SSRC Quarterly Econometric Model of the United States* (Chicago: Rand McNally and Co. and Amsterdam: North-Holland Publishing Co., 1966).

[32] The 86–order 1958 Input-Output Table for the United States has already been aggregated into a 33–sector table, as part of the Brookings Econometric Model Project, to be used in a study of the relationship between GNP expenditure components and industry outputs. See Michael D. McCarthy, "On the Aggregation of the 1958 Direct Requirements Input-Output Table," *The Review of Economics and Statistics,* XLIX, 4 (November, 1967), 600–602.

experiments should help sharpen our current understanding of the sensitivity of input-output predictions to alternative model structures and empirical specifications, and should provide further insights into the sensitivity of input-output structures to levels of aggregation. In this way, the research reported here should prove to be of value in constructing and using input-output systems, particularly as part of *mixed* models.

C. THE STRUCTURE OF THE BASIC MODEL

The discussion should probably start with a few clarifications. In reference to input-output models, *static* models are those that depict the interindustry flows of goods and services in current accounts and that comprise only one structural matrix, excluding all interindustry transactions in capital accounts. Dynamic models, on the other hand, include interindustry transactions carried on both in current and in capital accounts (i.e., they have two structural matrices, one for direct input requirements and another for capital requirements) where *time* is explicitly introduced into the system.

Static models are divided into *open* and *closed* types. In an *open* system, as first developed by Leontief, final consumption goods in the economy (i.e., final consumption by households, government expenditures, exports, and investment needs) are stipulated exogenously, and, thus, are not explained by the system of structural relationships characterizing the model. In a *closed* system, however, final consumption goods are made endogenous to the model. That is to say, all demands are explained by the system itself. In mathematical terms, *open* models are expressed as a system of nonhomogeneous linear equations, whereas *closed* models are expressed as a system of homogeneous linear equations.

In view of the development of *mixed* models, the traditional distinctions between dynamic and static, or open and closed input-output models are no longer as important, in operational terms,as they used to be. The *mixed* models take many forms, but basically they use the *open* system and make the model dynamic in ways different from that originally formulated by Leontief.

Like other models, the basic open model presents a simplified view of reality. It depicts the industrial *whirlpools* or patterns of circular economic interdependence as a simultaneous production system in which no stage of production depends upon definable *previous* stages. Everything is needed to produce everything, and unlike in the Austrian school of economic thought, there are no early stages or late stages. Coal is required for the production of steel and steel is required for the production of automobiles. Neither precedes the other. "Simultaneity is the mathematical economist's way of cutting through circular interdependence and avoiding all infinite-series multiplier chains."[33]

It is now widely known that the basic open model can be regarded as a special case of a linear programming or activity-analysis model.[34] The basic model thus refers to a production system, and is concerned with the functional interrelationships arising from production. In its simplest form, the number of available production processes is equal to the number of commodities that can be produced, where each process yields a single commodity, or more correctly, a homogeneous group of commodities. In the model, the substitution of alternative inputs is technologically not feasible;[35] and the *optimizing* solution equivalent to a linear programming solution is the one and only efficient solution possible to the system of simultaneous linear equations characterizing the model.

1. Definitions

To comprehend the logic of the open-static input-output model, it is convenient to think of a system of interindustry flows of goods and services, in which the whole economy is broken down into mutually exclusive *activity* categories or *sectors* and where each sector appears twice, once as a supplier of goods and services and then again as a consumer. Thus, the input-output tableau resembles a double-entry bookkeeping system, or simply a matrix, in which every row stands for the supply of a certain group of goods and services and every column represents a consuming sector's input schedule. Each identity cell or diagonal element in such a matrix shows an intra-industry flow of goods and services.[36]

On the production or supply side, a distinction is made between *produced commodities* (or *primary products*) and *primary factors.* As we will see later when we discuss

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[33] Robert Dorfman, Paul A. Samuelson, and Robert M. Solow, *Linear Programming and Economic Analysis* (New York: McGraw-Hill Book Co., Inc., 1958), p. 235.

[34] *Ibid.,* pp. 210–215.

[35] This often misunderstood point refers to Samuelson's substitution theorem, which is discussed in Appendix B.

[36] In practice, it is sometimes assumed that an industry does not use any of its own goods as input in producing its particular output. This practice introduces a downward bias in the measurement of sectoral output levels and consequently leads to upward distortions in the derivation of the *direct input requirements* or *technological* coefficients matrix.

primary and *secondary* products of industrial sectors, this distinction between *produced commodities* (or *primary products*) and *primary factors* is often a confusing one. To avoid this confusion, it is perhaps helpful to think of them as *produced* and *primary* inputs. Even this distinction may not help much, especially in situations where imports are treated, as in the Dutch input-output tables, as part of what we have called *primary factors.* It is generally better, then, to think of *primary factors* as a combination of labor input (i.e., wages and salaries paid to households), capital depreciation allowances (i.e., cost of various *vintages* of capital inputs during the base period), tax payments to all levels of government (i.e., a surrogate for services rendered by government), and entrepreneurial income (i.e., entrepreneurial input). Collectively, they are often labeled *value added, charges against final product,* or *payments sector.*

Similarly, on the consumption side, a distinction is made between *intermediate* and *final* consumption. *Intermediate* use refers to the consumption of raw materials and semi-finished goods, as well as services, by the *processing sectors* which encompass all productive sectors in the economy. *Final* use, on the other hand, refers to the consumption of *finished* goods and services by the *final demand sectors,* which include households (i.e., purchases of goods and services for personal consumption), gross capital formation, government purchases of goods and services (all levels of government), and exports. Thus, while *intermediate demand* refers to the total use of a given good by all the processing sectors in the economy, *final demand* refers to the consumption of a given good by all the final demand sectors. The row-wise summation of *intermediate* and *final demand* entries, then, equals the total output of goods and services by each producing sector. The vector of goods and services from all producing sectors required for consumption by the final demand sectors is often called the *final bill of goods* or simply *bill of goods.* This resembles the shopping list of finished goods and services required from the production system for final consumption.

Ignoring imports and joint products, we can now bring together the various elements of the accounting scheme underlying the basic open model, as shown in Figure 1.

FIGURE 1

INTERINDUSTRY FLOW OF GOODS AND SERVICES

Inputs from ↓ / Inputs to →		Consuming Sectors: Intermediate Use (Processing Sectors)	Consuming Sectors: Final Use (Final Demand Sectors)	Total Output
Producing Sectors	Produced Commodities (Inputs)	x_{11} x_{12} ... x_{1n} x_{21} x_{22} ... x_{2n} . . . . x_{n1} x_{n2} ... x_{nn}	y_{11} y_{12} ... y_{1p} y_{21} y_{22} ... y_{2p} . . . . y_{n1} y_{n2} ... y_{np}	x_1 x_2 . x_n
Producing Sectors	Primary Factors (Inputs)	w_{11} w_{12} ... w_{1n} w_{21} w_{22} ... w_{2n} . . . . w_{m1} w_{m2} ... w_{mn}	v_{11} v_{12} ... v_{1p} v_{21} v_{22} ... v_{2p} . . . . v_{m1} v_{m2} ... v_{mp}	w_1 w_2 . w_m
	Total Outlays	x_1 x_2 ... x_n	y_1 y_2 ... y_p	z

KEY:

x_{ij}: Value of product (service) of sector i consumed as an intermediate good by sector j in order to produce sector j's output.

w_{kj}: Value of primary factor k (e.g., labor, capital depreciation, etc.) consumed by sector j in order to produce sector j's output.

y_{ih}: Value of product (service)of sector i delivered to final demand sector h for final consumption.

v_{kh}: Value of primary factor k consumed by final demand sector h.

The table can also be expressed as a matrix, partitioned into four sub-matrices. It will then take the following form:[37]

$$\begin{matrix} [x_{ij}] & [y_{ih}] \\ [w_{ki}] & [v_{kh}] \end{matrix}$$

Each of these four sub-matrices refers to a different type of use:

$[x_{ij}]$ is of the order nxn and states the flow of goods and services, as inputs, from the producing sectors to the consuming sectors within the production system, and thus comprises the *interindustry transactions matrix.*

$[w_{kj}]$ is of the order m x n and states the input of primary factors in the production system.

$[y_{ih}]$ is of the order n x p and states the input of produced commodities for final demand. Each row of this sub-matrix, when summed, states the value of final product of type i consumed by the final demand sectors.

$[v_{kh}]$ finally, is of the order n x h, and states the consumption of primary factor inputs for final demand.

While the first three sub-matrices are directly connected within the production system (i.e., $[x_{ij}]$ and $[w_{kj}]$ stating the use of produced commodities and primary factor inputs, respectively, and $[y_{ih}]$ stating the final consumption of *finished* goods produced within the system), $[v_{kh}]$ represents flows not connected with activity within the production system, either on the production or the consumption side.[38]

2. Measurement of Intersectoral Flows

Intersectoral flow of goods and services in the input-output system are expressed in monetary value terms (e.g., in dollars, yens, etc.), based on average prices prevailing during the base period for which the input-output system has been constructed. Early in his work, Leontief used dollar values only, and most model-builders have ever since followed him in this practice.

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[37] A similar treatment can be found in Bengt Höglund and Lars Werin, *The Production System of the Swedish Economy, An Input-Output System,* Stockholm Economic Studies, New Series, IV (Stockholm: Almqvist and Wiksells, 1964), p. 18.

[38] *Ibid.*

In concept, at least, intersectoral flow of goods and services can be expressed in physical rather than in monetary terms. Thus, flows can be expressed in dimensional units defined by *a dollar's worth.* That is, if eggs are 50 cents a dozen, we can use two-dozen eggs as the physical unit in which the flows can be expressed in a particular case.

Leontief has shown the use of monetary value data alone leads to certain paradoxes.[39] Two systems with exactly the same technological coefficients matrix may in fact be quite different. For example, the price level in one system may be higher than the price level in the other. One may use dollars and the other rubles. One may be one-tenth the scale of the other, such as a region within a country. The mix of output levels and the distribution of labor into different sectors may be different. One may be wealthy and the other very poor due to lower labor productivity. Different physical meanings might be attached to a monetary unit's worth of coal, for example. Thus, two systems with the same *shadow* may in fact be quite different.[40]

If intersectoral flows information were available in physical units, such as in tons, kilowatts, etc., the resulting input-output system would serve most purposes as well or perhaps better than its counterpart expressed in monetary values, since the interminable problems associated with price adjustment could thus be avoided. While the notion may thus seem attractive, its adoption would give rise perhaps to more problems than it would be presumed to have solved. For example, it would require the definition of each sector in the system in fairly disaggregated terms. Given, however, the fact that there is an enormous number of products and that the aggregation of individual products measured in different physical units would by itself create quite obviously insurmountable problems, the idea is not workable. Furthermore, in highly developed economies where the nonmanufacturing, particularly the service, sectors are growing to be quite dominant, using physical units as the basis for measurement in an input-output system would make little sense.

In short, while for some purposes it may be desirable to express intersectoral flow of goods and services in terms of physical units rather than in monetary values, it is apparent that this is fraught with both theoretical and empirical problems. It is also clear that these theoretical and empirical problems are far more bothersome than the *paradoxes* indicated earlier.

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[39] Leontief, *op. cit.,* pp. 45-65 *[The Structure . . .].*

[40] Dorfman *et al., op. cit.,* pp. 239-240.

3. Accounting Relationships and the Derivation of *Technological* Coefficients

The important accounting relationships in the system can be expressed as follows:

$$(1.1) \qquad x_j = \sum_{i=1}^{n} x_{ij} + \sum_{k=1}^{n} w_{kj} \quad i = 1,\ldots,n; \; j = 1,\ldots,n; \; k = 1,\ldots,m.$$

where x_j = value of sector j's total inputs (outlays), which is equivalent to its total output.

$$(1.2) \qquad x_i = \sum_{j=1}^{n} x_{ij} + \sum_{h=1}^{p} y_{ih}$$

where x_i = total output of sector i. Thus, for i = j, we have

$$(1.3) \qquad x_i = x_j = (\sum_{j=1}^{n} x_{ij} + \sum_{h=1}^{p} y_{ih}) = (\sum_{i=1}^{n} x_{ij} + \sum_{k=1}^{m} w_{kj})$$

and

$$(1.4) \qquad \sum_{k=1}^{m} \sum_{i=1}^{n} w_{kj} = \sum_{i=1}^{n} \sum_{h=1}^{p} y_{ih} ,$$

which is the gross national product (gross regional product) identity, subject to the particular handling of imports in the system.

It is necessary to first develop an input-output table as shown in Figure 1, in order to derive *direct input coefficients* (or *technical, technological, production* coefficients). It should be noted again that of the four submatrices in the system, $[x_{ij}]$ and $[w_{kj}]$ refer to use within the production system. Thus, each column (1,...,n) represents the vector of inputs of a particular consuming sector, which is the same as that sector's cost vector. We have defined the production system here in such a fashion that, ignoring imports and joint-products, total costs of a given consuming sector is equal to the value of its total production. The direct input coefficients can be derived as follows:

$$(1.5) \qquad a_{ij} = \frac{x_{ij}}{x_j} = \frac{x_{ij}}{\sum_{i=1}^{n} x_{ij} + \sum_{k=1}^{m} w_{kj}}$$

where a_{ij} represents the quantity (value) of sector i's product that is required per unit of sector j's physical output (value of production). It is alternatively called the *technical, technological, production* or *direct input requirements* coefficients.[41] The matrix $A = [a_{ij}]$ is thus called the *technical, technological, production,* or *direct input* coefficients matrix.

In the input-output model the assumption is always made that each a_{ij} is positive or zero; that is, $a_{ij} \geqslant 0$. This follows from the definition of an input and from the postulate that each sector or elementary production process (activity) defined in the model produces only one, homogeneous *output.* In any case, the formal treatment of the input-output model in particular and of linear systems in general is made immensely complicated by allowing negative coefficients. It shall always be supposed here, therefore, that $a_{ij} \geqslant 0$ and that $A = [a_{ij}]$ is a nonnegative square matrix.

Primary factor input coefficients can likewise be derived as follows:

$$(1.6) \qquad b_{kj} = \frac{w_{kj}}{x_j} = \frac{w_{kj}}{\sum_{i=1}^{n} x_{ij} + \sum_{k=1}^{m} w_{kj}}$$

where b_{kj} represents the value of a given primary factor input (e.g., labor) required per unit of sector j's output.

Both (1.5) and (1.6) permit us to re-write (1.1) as follows:

$$(1.7) \qquad x_j = \sum_{i=1}^{n} a_{ij} \, x_i + \sum_{k=1}^{m} b_{kj} \, x_i \, .$$

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[41] Following the terminology originally used by Walras, early in his work Leontief called these *coefficients of production.* See Leontief, *op. cit.,* p. 37. *[The Structure . . .]*

From this we are led to:

$$\sum_{i=1}^{n} a_{ij} + \sum_{k=1}^{m} b_{kj} = 1, \tag{1.8}$$

which means that the sum of all direct input coefficients and primary factor input coefficients for each consuming sector (*activity, process*) is equal to 1.
Since

$$\sum_{k=1}^{m} b_{kj} > 0, \text{ then } \sum_{i=1}^{n} a_{ij} < 1.$$

This conclusion is rather important in the solution of the input-output system, as discussed in considerable detail in Appendix A.

4. Mathematical Formulation

It is not now too difficult to see that we can express the distribution of the output of each producing sector to all consuming sectors (i.e., to both the processing and the final demand sectors) as follows:

$$\begin{aligned} x_1 &= a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + y_1 \\ x_2 &= a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n + y_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= a_{n1} x_1 + a_{n2} x_2 + \ldots + a_{nn} x_n + y_n \end{aligned} \tag{1.9}$$

where

$$a_{ij} x_j = x_{ij}, \; x_i = \sum_{j=1}^{n} a_{ij} x_j + y_i$$

or since

$$y_i = \sum_{h=1}^{p} y_{ih}, \; x_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{h=1}^{p} y_{ih}.$$

This (1.9) can be re-written as follows:

$$(1.10)\qquad \begin{bmatrix} x_1 - a_{11}\, x_1 - a_{12}\, x_2 - \ldots - a_{1n}\, x_n \\ x_2 - a_{21}\, x_1 - a_{22}\, x_2 - \ldots - a_{2n}\, x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_n - a_{n1}\, x_1 - a_{n2}\, x_2 - \ldots - a_{nn}\, x_n \end{bmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

which can be reduced to the following form through simple factoring out of the identity entries:

$$(1.11)\qquad \begin{bmatrix} x_1\,(1 - a_{11}) - a_{12}\, x_2 - \ldots - a_{1n}\, x_n \\ - a_{21}\, x_1 + x_2\,(1 - a_{22}) - \ldots - a_{2n}\, x_n \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ - a_{n1}\, x_1 - a_{n2}\, x_2 - \ldots + x_n\,(1 - a_{nn}) \end{bmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

More compactly, we can write (1.11) as:

$$(1.12)\qquad \begin{bmatrix} (1 - a_{11}) & - a_{12} & - \ldots & - a_{1n} \\ - a_{21} + & (1 - a_{22}) & - \ldots & - a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ - a_{n1} & - a_{n2} & - \ldots + & (1 - a_{nn}) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

and finally as:

$$(1.13)\qquad (I - A)\, X = Y$$

where

I	:	the identity matrix,
A	:	the matrix of direct input coefficients, a_{ij},
X	:	the column vector of total sectoral output levels and,
Y	:	the column vector of final demand (all final demand sectors combined) for the *finished* products of all producing sectors (i.e., *final bill of goods*).

Here (I – A) is called the Leontief matrix, in which all the diagonal elements are positive while the off-diagonal elements are negative or zero. The equation system (1.13) can now be written in its usual, standard form as:

(1.14) $$X = (I - A)^{-1} Y$$

where $(I - A)^{-1}$ is the inverse of the Leontief matrix. This equation system (1.14) represents the generalized mathematical statement of the basic open input-output model. We should, therefore, point out its economic significance. It states that given a matrix of *technological* coefficients for an economy at a given point in time (i.e., A_t), and given also the column vector of exogenous *bill of goods* (i.e., Y_t) demanded from the production system for purposes of final consumption, we can determine the level of output that will be required from every sector in order to fulfill this exogenous demand, by explicitly taking into account the patterns of direct and indirect interdependence among the sectors of production in the economic system.

It should be noted here that $(I - A)^{-1} Y \neq Y (I - A)^{-1}$ since multiplication of matrices is not commutative, except in two very special instances: (i) multiplication involving a null matrix, that is if M_p is a square matrix of order p, and if 0_p is likewise with all elements zero, then $0_p M_p = M_p 0_p = 0_p$, and (ii) multiplication involving a diagonal matrix having all diagonal elements equal to unity (i.e., an identity or unit matrix, denoted as I), where $I_p M_p = M_p I_p = M_p$.[42]

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[42] See, for example, S. R. Searle, *Matrix Algebra for the Biological Sciences (Including Applications in Statistics)* (New York: John Wiley and Sons, Inc., 1966), pp. 35–37.

Clearly, the first instance has no economic meaning, interpreted in terms of input-output models, since the economic system must provide some final consumption besides simply managing to keep itself going.[43]

The second instance, however, indicates that the equality $(I - A)^{-1}\ Y = Y\ (I - A)^{-1}$ holds only when Y is an identity matrix. This has special significance in input-output analysis, as explained in Appendix B, since it underlines the economic meaning of the inverse Leontief matrix.

5. Solution of the Basic Open Model

By a *solution* of the open input-output system we mean a set of nonnegative values for the unknowns of the system (i.e., value of sectoral output levels denoted by the column vector X), which, when substituted for these unknowns turns each equation into a numerical equality or an identity. Although each solution contains a value for each unknown, the whole set of values is considered to constitute a *single* solution,[44] which is alternatively called a *solution set.*[45] A solution to the basic open-static input-output equation system (1.13) or (1.14) consists of a nonnegative column vector, nxl, expressing the value of output that would be required from every sector in order to satisfy the exogenously stipulated final demand vector or *bill of goods.*

A set of nonhomogeneous linear equations BX = Y can be solved if, and only if, they are consistent (i.e., if, and only if, the rank of the augmented matrix [B,Y] equals the rank of B, r(B)). If B is a matrix of q columns and rank r, and if Y is a non-null vector, $Y = (y_i) \neq 0$, the number of linearly independent non-null (LINN) solutions to the consistent set of equations BX = Y is given by q − r + 1. Thus, in order for only *one* LINN solution vector to exist, the rank of the matrix B must equal the number of the columns (unknowns). That is, the

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[43] Robert Solow, "On the Structure of Linear Models," *Econometrica,* XX, 1 (January 1952), p. 31.

[44] B. E. Margulis, *Systems of Linear Equations,* translated and adapted from the Russian by Jerome Kristian and Daniel A. Levine (New York: The Macmillan Co., 1964), p. 12.

[45] Ben Noble, *Applied Linear Algebra* (Preliminary Edition; Englewood-Cliffs, N. J.: Prentice-Hall, Inc., 1966), p. 76.

rank of B must equal its order (i.e., B must be of full rank). In other words, the columns (rows) of B must be linearly independent. This will be true only if the determinant of B is non-zero (i.e., B is a nonsingular matrix). The nonsingularity of B guarantees the existence of an inverse, B^{-1}. Thus, in summary, a set of linear nonhomogeneous equations $BX = Y$ has a unique non-null solution $X = B^{-1}Y$, for a non-null vector of constants, Y, if the inverse of B, B^{-1}, exists, where B is a non-null square matrix.

In an open input-output system all components of the solution vector $X = (x_i)$ must be nonnegative, corresponding to a nonnegative final demand vector.That is, every element of the Leontief inverse, $(I - A)^{-1}$ must be nonnegative (i.e., $\alpha_{ij} \geqslant 0$). This nonnegativity restriction on the solution vector sets the basic input-output system apart from the standard linear nonhomogeneous equations systems, since in the latter, negative solutions and solutions with negative components are admissible. The mathematical conditions for the existence and uniqueness of a nonnegative solution set for the open Leontief system are discussed in considerable depth in Appendix A. It can be quite safely said that the material presented in Appendix A on the mathematical properties of the open Leontief system represents perhaps the most exhaustive and unified treatment of the subject yet to appear in input-output literature.

There are a number of methods that can be used to solve the open input-output system. Direct methods (i.e., those that would lead to the true solution of the given system if all computations were carried out without roundoff) involve variations of the elimination procedure, such as Gaussian elimination or Gauss-Jordan elimination. Alternatively, iterative methods, particularly the Gauss-Seidel iterative procedure (also referred to as pointwise relaxation), can be used. Further, the system can always be solved through matrix inversion, by applying Cramer's rule, if the Leontief matrix is nonsingular.

If the Leontief inverse is required, as it often is in actual input-output practice, the Gauss-Jordan method is generally used.[46] It is also useful to know that, for reasons explained in considerable detail in Appendix A, the power series expansion $I + A + A^2 + \ldots + A^p$ converges to the Leontief inverse $(I - A)^{-1}$.

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[46]Clopper Almon, Jr., *Matrix Methods in Economics* (Reading, Mass.: Addison-Wesley Publishing Co., 1967), p. 29. Also included here is a discussion of the Gauss-Jordan and iterative methods (pp. 1–30).

A mathematical discussion of these respective methods can be found in numerous texts, such as those by Dorn and Greenberg,[47] Faddeev and Faddeeva,[48] James, Smith, and Wolford,[49] Householder,[50] John,[51] Margulis,[52] Norkin,[53] Ralston,[54] Varga,[55] and Vilenkin.[56] The reason for listing so many sources is that, ignoring plenty of repetition which is inevitable, they complement one another in coverage, interpretation, and level of mathematical treatment.

The elimination method of solving the open input-output system yields sufficiently accurate solutions for as many as 15 to 20 equations, the exact number depending on the actual equations, the roundoff procedure followed, and the number of digits retained in the results

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[47] William S. Dorn and Herbert J. Greenberg, *Mathematics and Computing: With FORTRAN Programming* (New York: John Wiley and Sons, Inc., 1967), pp. 299–323.

[48] D. K. Faddeev and V. N. Faddeeva, *Computational Methods of Linear Algebra,* trans. by Robert C. Williams (San Francisco: W. H. Freeman and Co., 1963).

[49] M. L. James, G. M. Smith, and J. C. Wolford, *Applied Numerical Methods for Digital Computation with FORTRAN* (Scranton, Penn.: International Textbook Co., 1967), pp. 184–236.

[50] A. S. Householder, *The Theory of Matrices in Numerical Analysis* (New York: Blaisdell Publishing Co., A Division of Ginn and Co., First Edition, 1964), pp. 91–121.

[51] Fritz John, *Lectures on Advanced Numerical Analysis* (New York: Gordon and Breach, Science Publishers, Inc., 1967), pp. 1–35.

[52] Margulis, *op. cit.,* pp. 11–33.

[53] S. B. Norkin, *The Elements of Computational Mathematics,* translated by G. J. Tee and English translation edited by A. D. Booth (New York: The Macmillan Co., 1965), pp. 89–114.

[54] Anthony Ralston, *A First Course in Numerical Analysis* (New York: McGraw-Hill Book Co., Inc., 1965), pp. 394–463.

[55] Richard S. Varga, *Matrix Iterative Analysis* (Englewood Cliffs, N. J.: Prentice-Hall, Inc., Third Printing, 1965).

[56] N. Ya. Vilenkin, *Successive Approximation,* translated and adapted from the Russian by M. B. P. Slater and J. W. Teller (New York: The Macmillan Co., 1964), pp. 60–63.

of the arithmetic operations.[57] By using roundoff error control procedures (e.g., error equations, double precision arithmetic) the number of equations that can be handled can be raised considerably higher.

Among the iterative methods, the Gauss-Seidel method[58] is extremely well suited to the task in input-output analysis, provided that the Leontief inverse is not desired. The Gauss-Seidel method generally has the disadvantage of not always converging to a solution and of sometimes converging very slowly when it does converge. However, this disadvantage does not seem to appear in input-output applications, since the mathematical conditions for the convergence of the Gauss-Seidel method are partly identical with certain mathematical properties of the open input-output system [refer to Appendix A] that guarantee the existence and uniqueness of a nonnegative solution.

In order for the Gauss-Seidel method to converge to a solution, basically two conditions must be met. First, the coefficients matrix must be *diagonally dominant,* that is, the sum of the absolute values of the coefficients in each row must be less than or equal to the absolute value of the diagonal element in the same row, such that at least one of the inequalities thus formed is not an equality. This is the input-output dominant diagonality condition discussed in Appendix A. Secondly, the coefficients matrix must be irreducible (indecomposable).[59] As the mathematical discussion given in Appendix A should indicate, there is no reason to expect the *technological* coefficients matrix (referred to as the Minkowski-Leontief matrix in Appendix A) to be indecomposable. This means that, as long as the *technological* coefficients matrix is indecomposable, convergence of the Gauss-Seidel method should pose no problem. Further, if the *technological* coefficients matrix is *sparse* (i.e., it has a large number of zeroes), this will cut down on the number of iterations and the work needed to obtain a solution will be greatly reduced.[60]

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[57] James, Smith, and Wolford, *op. cit.,* p. 218.

[58] As a minor but interesting point, Householder quotes Forsythe to have remarked that the Gauss-Seidel method was not known to Gauss (although he did use a method of relaxation) and it was not recommended by Seidel. See Householder, *op. cit.,* p. 115.

[59] Dorn and Greenberg, *op. cit.,* p. 313.

[60] *Ibid.*

Finally, it should be mentioned that the roundoff error is generally less serious in the Gauss-Jordan iteration (and for iterative methods in general) than it is for direct methods. However, the roundoff error can be quite serious when the *technological* coefficients matrix is *ill-conditioned.*[61]

6. The Relationship Between Final Demand and Interindustry (Intermediate) Demand

The equation system (1.13) or (1.14), describing the basic open input-output model, expresses the relationship between the exogenously specified final demand vector and the total *equilibrium* sectoral output levels that will be required to satisfy this exogenous demand. Since final consumption demand is exogenously determined, the basic open model actually *predicts* or *explains* total interindustry (intermediate) demand for goods and services, when we remember that each sector's total output is equal to its production sold to all industries in the system for intermediate consumption *plus* its shipments of finished goods to the final demand sectors for final consumption. The basic model can be easily reformulated, to show the relationship between final demand and interindustry demand for goods and services.

We can start with the identity $X = AX + Y$ which is simply a more compact statement of (1.9), where AX stands for interindustry demand and Y for final consumption demand. We can let $AX = Z$, where now Z represents the interindustry demand vector. Since, from (1.14), $X = (I - A)^{-1} Y$, we can write the identity $X = AX + Y$ as follows:

$$(I - A)^{-1} Y = Z + Y \tag{1.15}$$

which can then be solved for Z as

$$Z = (I - A)^{-1} Y - Y \tag{1.16}$$

which, in more compact notation, can be written as

$$Z = [(I - A)^{-1} - I] Y. \tag{1.17}$$

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[61] Ralston, *op. cit.*, p. 233. A matrix is *ill-conditioned* if, when it has been normalized so that its largest element has order of magnitude equal to unity, its inverse has very large elements [pp. 233, 396-397]. The danger inherent in solving an *ill-conditioned* system stems from the fact that a small change or error in an element of the coefficient matrix can cause a large change or error in the solution.

Introducing appropriate *time* subscripts, we now have

$$Z_{t+\tau} = [(I - A_t)^{-1} - I]\ Y_{t+\tau} \tag{1.18}$$

where t refers to a given base year and $(t + \tau)$ to a future year $(\tau = 1,2,...T)$. The system can be used, under the assumption of constant *technological* coefficients, to obtain conditional point predictions of interindustry demand for goods and services, by exogenously stipulating the final demand vector for the target year, expressed in base year prices.

7. Related Topics in Input-Output Analysis

A *complete* discussion of input-output analysis would certainly cover such additional topics as the following: (a) the inverse of the Leontief matrix and *matrix multipliers,* (b) the relationship between final demand, sectoral output requirements, and value added (income generation), (c) the substitution theorem, (d) the Leontief Paradox, (e) prices in the open input-output system, (f) the closed input-output model, (g) the dynamic input-output model and (h) the relationship of the basic open model to linear programming. Since a discussion of these topics here would unduly hinder the continuity of the text in this chapter, they are covered in Appendix B, in keeping with a general effort made in this dissertation to make it as self-contained as possible.

D. CRITICAL INTERPRETATIONS OF THE BASIC MODEL AND THE FOCUS OF THIS DISSERTATION

In the past, the basic model discussed here has been criticized by many writers. Two excellent critical reviews, for example, have been given by Dorfman[62] and Hurwicz.[63] Most of these earlier criticisms have generally focused on five major areas: (1) the key assumptions of the basic open model, (2) important variables that are either omitted or relegated to the background, (3) the interpretation of the *technological* coefficients matrix, (4) the treatment of *time*, and (5) the statistical properties of the model. It is instructive to examine these major criticisms, which have been raised more than a decade ago, from the perspective of the present. Such an examination should also serve as a context in which to place the subject of this dissertation.

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[62] Robert Dorfman, "The Nature and Significance of Input-Output," *The Review of Economics and Statistics,* XXXIV, 2 (May, 1954), 121–133.

[63] Leonid Hurwicz, "Input-Output Analysis and Economic Structure: A Review Article," *The American Economic Review,* XLV, 4 (September, 1955), 626–636.

First, the model has come under attack for a few key assumptions on which it is based. These key assumptions can be summarized as follows:

(1) Each production *process* leads to the production of *one* commodity, and only *one* process corresponds to each produced commodity.

(2) The quantity of each consumed commodity is related to the quantity (i.e., output) of each produced commodity in terms of a linear function; that is, the production function for each process is linear homogeneous of degree one, with constant returns to scale (i.e., there are no economies of scale in the system). Furthermore, the proportionality coefficients, represented by the *technological* coefficients, are *fixed.*

(3) There is additivity between the various processes in the system.

These assumptions now cause nowhere as much distress as they perhaps did more than a decade ago. To begin with, the seeming rigidity imposed by the first assumption can now be relaxed, by having a rectangular A matrix. How this can be done is explained in the next chapter.

As to the linear homogeneous production function and additivity assumptions, it should be noted that they are now widely accepted as part of a general theoretical structure that has emerged during the past two decades with the development of generalized activity models. The traditional distinction between the classical marginal productivity approach to the theory of production and the fixed-coefficient approach is not now as serious as it used to be, as the two have been synthesized in the development of activity models, such as linear programming. The marginal productivity theory, in which great stress is put on the existence of alternative methods of producing the same product and on the choice among them on the basis of profit maximization, may be appropriate for long-run choices, where the quantities and types of capital may be altered. The fixed-coefficient theory, on the other hand, may be more useful in the study of short-run choices, where the different types of production processes available are severely limited, given the stock of capital goods. At any point in time, the relations between inputs and outputs are of the fixed-coefficient type. This does not rule out the possibility that the coefficients will change over time in response to the changes in fixed capital structure.

Finally, the assumption of *fixed* or temporally invariant *technological* coefficients now present no special problems. The experience in this country and abroad in recent years shows that this problem can be overcome in two ways. First, an existing input-output model can be kept up-to-date by applying a variety of updating methods, such as the linear programming technique developed by Matuszewski, Pitts, and Sawyer[64] or the RAS method developed by Richard Stone and his associates.[65] Secondly, with the advent of the large-scale *mixed* forecasting models, as mentioned earlier, there has developed in recent years the capability to monitor the causes and effects of changes in the *technological* coefficients and to forecast changes in at least the key coefficients. Industry specialists have made a substantial contribution in this respect and are likely to play a greater role in the future.

Secondly, the open model has been criticized for omitting both consumption and investment, the two driving forces of the economy. Further, it is felt, considerations of profit maximization, consumer utility maximization, optimal allocation of resources, and motivation occur in the model only in the background, if at all, while the foreground is preoccupied by production relations. Apart from questions concerning maximization, it is fair to say that such criticisms are no longer valid. Again, with the development of *mixed* models, investment and consumption, as well as other variables, are explicitly taken into account by using appropriate econometric methods.

Thirdly, the interpretation of the *technological* coefficients matrix has been a point of controversy. Over the years, Leontief and his associates have called these coefficients *structural* coefficients. Although the term *structural* indicates considerable amount of invariance and, in fact, the coefficients have been explicitly assumed to remain constant over time, changes in them have been interpreted by Leontief and others in terms of technological change in the economy. In a recent article, for example, Carter introduces her subject as

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[64] T. I. Matuszewski, P. R. Pitts, and John A. Sawyer, "Linear Programming Estimates of Changes in Input Coefficients," *The Canadian Journal of Economics and Political Science,* XXX, 2 (May, 1964), 203–210.

[65] Stone, *loc. cit.;* Stone and Brown, *loc. cit. [Supra, footnote 28].* For a well known application of the RAS method, see J. Paelinck and J. Waelbroeck, "Etude empirique sur l'évolution de coefficients 'input-output'," *Economie Appliquée,* XIV (January-March, 1963), 81-111.

follows: "this paper presents a comprehensive picture of technological change in the United States as it emerges from a systematic comparison of the 1947 and 1958 input-output tables."[66] The opposing school of thought has pointed out that traditional economics does not require the hypothesis of technological change alone to account for the change in the *technological* coefficients. Given the level of aggregation inherent in the model, it has been argued, it is perfectly conceivable that the coefficients change as a direct consequence of changes in the *product mix* of each consuming industry. It has been maintained, furthermore, that changes in relative prices need not derive from any technological change. In short, it is not clear, to many, even to this day, as to why changes in the A matrix should be interpreted as technological change. For this reason, throughout this dissertation, the first word of the term *technological* coefficients is written in italics to indicate a qualification in its interpretation.

Fourthly, the basic model has been criticized for its neglect of the time dimension, or more accurately, for the fact that it abstracts from the time sequence of production and interindustry transactions and, therefore, it applies only to a stationary equilibrium where time is of no consequence. To put it somewhat differently, predictions of sectoral output levels that can be obtained by using the open model refer only to equilibrium levels, without specifying how long it will take for the system to achieve this equilibrium or whether such equilibrium will be achieved at all. At least implicitly, the model operates on the assumption that in response to a shift in final demand in a given year, all the direct and indirect interindustry demands will be satisfied within the same year and that the predicted sectoral output levels refer to that particular year. Actually, it might take longer than a year for the system to satisfy all direct and indirect interindustry demands. For this reason, the accuracy of input-output predictions must be carefully qualified in respect to the time period to which the predicted production levels refer. The problem of time in input-output analysis can be solved by introducing time-lags into the system. Despite some past attempts, this area clearly offers an opportunity for further research.

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[66] Anne P. Carter, "Changes in the Structure of the American Economy, 1947 to 1958 and 1962," *The Review of Economics and Statistics,* XLIX, 2 (May, 1967), 209.

In the *fifth* place, the basic model has been criticized on statistical grounds. The *structural* parameters of the model are estimated through the use of one-observation samples. Consequently, the basic model suffers from the *degrees of freedom* problem. Usually, when an economic relation is fitted to observations, the general form of the relationship is stipulated beforehand, while the constants of the relationship or the parameters are estimated from the observations. The number of degrees of freedom is the excess of the number of observations over the number of parameters to be estimated, and it is positive when there are more observations than parameters. When there are as many observations as parameters, it is usually possible to choose the values of the parameters so that every observation exactly satisfies the fitted relationship. If there are more parameters than observations, then there are many possible ways of choosing the parameters so that the relationship satisfies every observation and there is no unique way of estimating the parameters. In economic analysis, there is usually no way of fitting the relationship to the observations exactly; that is, deviations usually occur and are ascribed to random variation due to explanatory variables not included in the analysis. The accuracy of the estimated relationship, in terms of statistical variability, increases with the number of degrees of freedom.[67]

The number of observations is severely limited in input-output analysis. To minimize the *degrees of freedom* problem, therefore, such strong assumptions must be made about the nature of the production relations that even a single observation will suffice to yield estimates. In the input-output case, the estimate of each parameter, based on single observation, has only zero degrees of freedom. This means that although there is a best single estimate of a given parameter, there is no way of estimating the reliability of that estimate. Thus, if a given observation is a random drawing from a normally distributed population and we wish to estimate the mean of the distribution, the observation itself certainly represents an appropriate estimate of the mean, even though the variance of the distribution, which is the same as the variance of the estimate about the true mean, cannot be estimated.[68]

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[67]See Kenneth J. Arrow and Marvin Hoffenberg, *A Time Series Analysis of Interindustry Demands* (Amsterdam: North-Holland Publishing Co., 1959), pp. 14–15.

[68]*Ibid.*

Among the five major areas of criticism examined above, the subject of this dissertation is related most closely to the fifth or last area concerning the statistical properties of the model. As noted earlier, the estimate of each parameter in the model is based on a single observation. Since so much depends on the method actually used in the measurement of each parameter, an attempt is made in this dissertation to conduct a series of sensitivity experiments to ascertain how input-output predictions are affected by alternative ways of estimating the model's parameters. In addition, an effort is made to ascertain whether or not input-output prediction errors vary systematically with a set of variables that describe certain fundamental empirical characteristics of the model. Thirdly, the focus is slightly shifted to find out to what extent input-output prediction errors that are directly attributable to aggregation (i.e., through the *aggregiation bias*) show systematic variations that convey an understanding of the *information loss* due to aggregation in alternative model formulations. Lastly, and following these experiments, a systematic mathematical exploration is made of alternative model structures for national, regional, and multiregional economic analysis, under alternative assumptions regarding the availability of certain types of information necessary in input-output model construction.

As the next two chapters should demonstrate, the areas of investigation that comprise the subject of this dissertation have long been neglected in the input-output literature. The results achieved here are, therefore, hoped to sharpen the general understanding of the mathematical, as well as empirical,structure of input-output models and their uses in national, regional, and multiregional economic analysis

E. EFFECTS OF MEASUREMENT ERRORS ON INPUT-OUTPUT PREDICTIONS

For a variety of reasons explained in the next chapter, an input-output model is subject to measurement errors. Theoretically, it is possible to decompose the observed final demand and sectoral output levels, as well as the intersectoral flows, into *true* values and measurement errors (i.e., perturbation). Indicating true values by – and measurement errors by ~, we can write the final demand vector as $y_t = \bar{y}_t + \tilde{y}_t$ and the sectoral output vector as $x_t = \bar{x}_t + \tilde{x}_t$. The *technological* coefficients can then be written as

(1.19)

$$a_{ij} = \frac{\bar{x}_{ij} + \tilde{x}_{ij}}{\bar{x}_j + \tilde{x}_j} = \frac{\dfrac{\bar{x}_{ij}\,(\bar{x}_{ij} + \tilde{x}_{ij})}{\bar{x}_{ij}}}{\dfrac{\bar{x}_j\,(\bar{x}_j + \tilde{x}_j)}{\bar{x}_j}} = \frac{\bar{x}_{ij}}{\bar{x}_j}\;\frac{1 + \dfrac{\tilde{x}_{ij}}{\bar{x}_{ij}}}{1 + \dfrac{\tilde{x}_j}{\bar{x}_j}} \approx$$

$$= \bar{a}_{ij}\,\frac{1 + \dfrac{\tilde{x}_{ij}}{\bar{x}_{ij}}}{1 + \dfrac{\tilde{x}_{ij}}{\bar{x}_j}} \approx a_{ij} + \bar{a}_{ij}F.$$

This equation indicates that the observed *technological* coefficients matrix can be decomposed as $A_t = \bar{A}_t + \tilde{A}_t$. It then easily follows that the (i,j)th element in the error matrix $\tilde{A}_t$ is equal to

$$(1.20) \qquad \tilde{a}_{ij} = a_{ij} - \bar{a}_{ij} = \bar{a}_{ij}\,\frac{1 + \dfrac{\tilde{x}_{ij}}{\bar{x}_{ij}}}{1 + \dfrac{\tilde{x}_j}{\bar{x}_j}} - \bar{a}_{ij} = \bar{a}_{ij}\left(\frac{1 + \dfrac{\tilde{x}_{ij}}{\bar{x}_{ij}}}{1 + \dfrac{\tilde{x}_j}{\bar{x}_j}} - 1\right).$$

If we let

$$(1.21) \qquad \mu = \frac{\dfrac{\tilde{x}_{ij}}{\bar{x}_{ij}}}{\dfrac{\tilde{x}_j}{\bar{x}_j}}$$

we can see immediately that $\tilde{a}_{ij}$ is zero when $\mu = 1$, positive when $\mu > 1$, and negative when $\mu < 1$.

We can now proceed to examine the inverse of a *perturbed* Leontief matrix. Ignoring the time subscripts temporarily, we can start by first establishing the following identity:

$$(1.22) \qquad [I - (\bar{A} + \tilde{A})] = (I - \bar{A} - \tilde{A}) = (I - \bar{A}) - (I - \bar{A})(I - \bar{A})^{-1}\tilde{A}$$

which can be re-written as

$$(1.23) \qquad (I - \bar{A} - \tilde{A}) = (I - \bar{A}) \; [I - (I - \bar{A})^{-1} \tilde{A}].$$

Using the well known fact in matrix algebra that the inverse of a product is the product of the inverses taken in reverse order, we have

$$(1.24) \qquad (I - \bar{A} - \tilde{A})^{-1} = [I - (I - \bar{A})^{-1} \tilde{A}]^{-1} \; (I - \bar{A})^{-1}.$$

This expression can be made clearer through a series of manipulations.[69]
First, it can be re-written as

$$(1.25) \qquad [I - (I - \bar{A})^{-1} \tilde{A}] \; (I - \bar{A} - \tilde{A})^{-1} = (I - \bar{A})^{-1}$$

which, when expanded on the left side, becomes

$$(1.26) \qquad (I - \bar{A} - \tilde{A})^{-1} - (I - \bar{A})^{-1} \tilde{A} (I - \bar{A} - \tilde{A})^{-1} = (I - \bar{A})^{-1}.$$

This is equivalent to

$$(1.27) \qquad (I - \bar{A} - \tilde{A})^{-1} = (I - \bar{A})^{-1} + (I - \bar{A})^{-1} \tilde{A} (I - \bar{A} - \tilde{A})^{-1}.$$

Finally, we obtain, through substitution,

$$\begin{aligned}(1.28) \qquad (I - \bar{A} - \tilde{A})^{-1} &= (I - \bar{A})^{-1} + (I - \bar{A})^{-1} \tilde{A} [(I - \bar{A})^{-1} \\ &\qquad + (I - \bar{A})^{-1} \tilde{A} (I - \bar{A} - \tilde{A})^{-1}] \\ &\approx (I - \bar{A})^{-1} + (I - \bar{A})^{-1} \tilde{A} \; (I - \bar{A})^{-1}.\end{aligned}$$

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[69]The same result can also be found in Theil and Tilanus, but they skip all the intermediate steps in their treatment. See Henri Theil, *Applied Economic Forecasting* (Amsterdam: North-Holland Publishing Co., and Chicago: Rand McNally and Co., 1966), p. 213; and C. B. Tilanus, *Input-Output Experiments, The Netherlands, 1948–1961* (Rotterdam: Rotterdam University Press, 1966), p. 103.

Since the exogenously determined final demand vector for a prediction year will also contain measurement errors, the input-output predictions will contain an error vector which would be a linear combination of the measurement errors inherent in both the final demand vector and the *technological* coefficients matrix:

$$
\begin{aligned}
(1.29) \qquad X^p_{t+\tau} &= [(I-\bar{A}_t)^{-1} + (I-\bar{A}_t)^{-1}\,\tilde{A}_t\,(I-\bar{A}_t)^{-1}]\;(\bar{Y}_{t+\tau}+\tilde{Y}_{t+\tau}) \\
&= (I-\bar{A}_t)^{-1}\,(\bar{Y}_{t+\tau}+\tilde{Y}_{t+\tau}) + (I-\bar{A}_t)^{-1}\,\tilde{A}_t\,(I-\bar{A}_t)^{-1}\,(\bar{Y}_{t+\tau}+\tilde{Y}_{t+\tau}) \\
&= (I-\bar{A}_t)^{-1}\;\bar{Y}_{t+\tau} + (I-\bar{A}_t)^{-1}\;\tilde{Y}_{t+\tau}) \\
&\qquad + (I-\bar{A}_t)^{-1}\;\tilde{A}_t\,(I-\bar{A}_t)^{-1}\,(\bar{Y}_{t+\tau} + \tilde{Y}_{t+\tau})
\end{aligned}
$$

where $X^p_{t+\tau}$ is the predicted vector of output levels, containing measurement errors, and where $(I-\bar{A}_t)^{-1}\,\bar{Y}_{t+\tau}$ is the *true* prediction of the output vector $\bar{X}^p_{t+\tau}$. The effects of measurement errors on input-output predictions can be finally summarized as follows (i.e., Eq. (1.29) minus $\bar{X}^p_{t+\tau}$):

$$
\begin{aligned}
(1.30) \qquad X^p_{t+\tau} - \bar{X}^p_{t+\tau} &= [(I-\bar{A}_t)^{-1} + (I-\bar{A}_t)^{-1}\,\tilde{A}_t\,(I-\bar{A}_t)^{-1}]\tilde{Y}_{t+\tau} \\
&\qquad + (I-\bar{A}_t)^{-1}\,\tilde{A}_t\,(I-\bar{A}_t)^{-1}\,\bar{Y}_{t+\tau}
\end{aligned}
$$

In summary, the measurement error component of the prediction vector $X^p_{t+\tau}$ is equal to a linear combination of the measurement errors that are inherent both in the *technological* coefficients matrix $(\tilde{A}_t)$ in year t and in the final demand vector $(\tilde{Y}_{t+\tau})$ in year $t+\tau$. The result is also applicable to the base year t, if we simply replace $t+\tau$ by t. Of course, if we assume that the final demand vector is observed without any error, then the measurement error component of the prediction vector consists of

$$
(1.31) \qquad X^p_{t+\tau} - \bar{X}^p_{t+\tau} = (I-\bar{A}_t)^{-1}\,\tilde{A}_t\,(I-\bar{A}_t)^{-1}\,\bar{Y}_{t+\tau}.
$$

In recent years, considerable attention has been given to the investigation of the effect of measurement and rounding errors on the numerical solution of linear algebraic systems. A list of some of the more important mathematical literature on this subject would be prohibitively long. An excellent discussion on this subject, with an exhaustive list of references can be found in an article by Albasiny.[70]

Errors in the input-output system can also be studied by assuming that the errors have stochastic properties. The input-output model can be viewed as a probabilistic system in which the elements of the solution vector are implicit random variables due to the stochastic nature of the *technological* coefficients matrix and, by extension, of the Leontief inverse. Of course, it is also possible to treat the output vector as a random vector, due to the stochastic properties of the final demand vector. Of particular interest in the literature is the work of Quandt,[71] who has studied the properties of probabilistic Leontief systems, given exact knowledge of the *true* Leontief matrix and the error variances and assuming that the errors in the *technological* coefficients matrix are independently and normally distributed. Etherington[72] has derived expressions for the moments of the errors in the solution vector for a system of simultaneous equations, under the assumption that the errors in the matrix of coefficients are symmetrical. Box and Hunter[73] have examined the development of confidence regions for the solution of sets of linear equations.

Error analysis cannot be conducted in input-output analysis insofar as the parameters are estimated on the basis of a single observation. As a substitute, however, attention can be focused on sensitivity experiments, examining, for example, the effects of controlled variations in the matrix of *technological* coefficients on the solution vector, and on input-output predictions. Such an approach is taken in this dissertation. Instead of introducing *controlled*

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[70] Ernest L. Albasiny, "Error in Digital Solution of Linear Problems," in Louis B. Rall (ed.), *Error in Digital Computation,* I (New York: John Wiley and Sons, Inc., 1965), 131–245.

[71] R. E. Quandt, "Probabilistic Errors in the Leontief System,"*Naval Research Logistics Quarterly,* Vol. V (1958), 155-170; and "On the Solution of Probabilistic Leontief Systems," *Naval Research Logistics Quarterly,* VI (1959), 295-305.

[72] I. M. H. Etherington, "On Errors in Determinants," *Proceedings of the Edinburgh Mathematical Society,* III (1932), 107–117.

[73] G. E. P. Box and J. S. Hunter, "A Confidence Region for the Solution of a Set of Simultaneous Equations with an Application to Experimental Design," *Biometrika,* XLI (1954), 190–199.

variations into the *technological* coefficients matrix, however, two separate *technological* coefficients matrices are used which differ from one another fundamentally in the way the individual coefficients are empirically estimated. The exact nature of these differences is explained fully in the next chapter.

F. THE AGGREGATION PROBLEM IN INPUT-OUTPUT ANALYSIS

To begin with, following Malinvaud,[74] it is necessary to draw a distinction between aggregation problems associated with empirical construction of input-output models, on the one hand, and problems associated with using input-output models, on the other. The concern here will be solely with the latter. The former, or the *sectorization* problem, is discussed in some detail in the next chapter. The discussion here, therefore, bypasses many problems connected with the construction of a model and the validity of that model in terms of the aggregation or sectorization procedures used.

The aggregation problem in input-output analysis, as it is viewed in the present context, arises when a previously compiled intersectoral flows or *technological* coefficients matrix is reduced into a smaller matrix by consolidating the sectors of the previously compiled matrix into groups of sectors. In input-output analysis, it is often desirable to reduce the size of a given matrix in order to increase its analytical manageability or to tailor the available matrix to the study of particular problems emphasizing, for example, certain *key* sectors. When sectors are combined into groups of sectors, the information value of the resulting smaller model is always less than or equal to that of the original model.[75]

Since the early days of input-output analysis, the aggregation problem has been a source of continual concern to many writers, as evidenced by the substantial body of literature on

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[74] Edmond Malinvaud, "Aggregation Problems in Input-Output Models," in Tibor Barna (ed.), *The Structural Interdependence of the Economy,* Proceedings of an International Conference on Input-Output Analysis, Varenna, 1954 New York: John Wiley and Sons, Inc., 1956), p. 190.

[75] Henri Theil and Pedro Uribe, "The Information Approach to the Aggregation of Input-Output Tables," *The Review of Economics and Statistics,* XLIV, 4 (November, 1967), 451.

the subject that has accumulated over the years.[76] Generally, two fundamental questions have been posed in the literature:

(a) what is the nature of the prediction error that is to be ascribed to aggregation (i.e., *aggregation bias*), if input-output predictions of output (or intermediate demand) levels are obtained by using a detailed (i.e., large) model on the one hand and a smaller (i.e., aggregated) model, on the other, when the prediction year is the same as the year for which the detailed input-output model has been constructed and when the prediction errors are

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[76] See, for example, the following (arranged in chronological order):

M. Hatanaka, "Note on Consolidation within a Leontief System," *Econometrica,* XX, 2 (April, 1952), 301–303;

J. B. Balderston and T. M. Whitin, "Aggregation in the Input-Output Model", in Oskar Morgenstern (ed.), *Economic Activity Analysis* (New York: John Wiley and Sons, Inc., 1954), pp. 79–128;

Malinvaud, *op. cit.,* 189–202;

M. McManus, "General Consistent Aggregation in Leontief Models," *Yorkshire Bulletin,* VIII (June, 1956), 28–48;

H. Theil, "Linear Aggregation in Input-Output Analysis", *Econometrica,* XXV, 1 (January, 1957), 111–122;

John McCarthy, "Aggregation in the Open Leontief Model," *Econometrica,* XXV, 4 (October, 1957), 602;

Walter D. Fisher, "Criteria for Aggregation in Input-Output Analysis", *The Review of Economics and Statistics,* XL, 3 (August, 1958), 250–260;

Kenjiro Ara, "The Aggregation Problem in Input-Output Analysis," *Econometrica,* XXVII, 2 (April, 1959), 257–262;

J. Skolka, *The Aggregation Problem in Input-Output Analysis* (Prague: Czechoslovakian Academy of Sciences, 1964);

V. Ginsburgh, "Critères théoriques et pratiques de l'agrégation dans input-output et validité de l'agrégation adoptée dans le modèle de croissance de Bruxelles," *Cahiers Economiques de Bruxelles,*25, 1[er] trimestre (1965), 106-123;

Theil and Uribe, *op. cit.,* 451–462;

Rose Mohr Rubin, "Aggregation Criteria in Input-Output Analysis" (unpublished Ph. D. Thesis, Kansas State University, 1968).

compared both at the global level (i.e., for all sectors as a whole) and at the level of the sectors as defined in the aggregated model, and

(b) what set of criteria can be developed for aggregation, so that the same results of analysis or predictions can be obtained by using an aggregated model as would be obtained by using a detailed model?

The problem of disaggregation, which is just the reverse of the problem represented by the latter of these two questions, has been discussed by Fei[77] and Malinvaud.[78] Given an input-output model, it is often desirable to gradually refine it into a much more detailed structure by following acceptable disaggregation procedures.[79] The problems of aggregation and disaggregation are related, but they are conceptually different problems both from the mathematical and from the economic point of view.

It appears from the literature on the aggregation problem that while a lot of emphasis has been put on the development of criteria for aggregation (e.g., the minimum distance criterion developed by Walter D. Fisher,[80] the rank criterion developed by John McCarthy,[81] and the information loss criterion developed by J. Skolka[82] and Theil[83]), too little attention

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[77] John C. H. Fei, "A Fundamental Theorem for the Aggregation Problem of Input-Output Analysis," *Econometrica,* XXIV, 4 (October, 1956), 400–412.

[78] Edmond Malinvaud, "L'agrégation dans les modéles économiques," *Cahiers du Seminaire d'Econométrie,* 4 (Paris, 1956), 69-146; particularly pages 113 and 139.

[79] According to Fei [see Fei, *op. cit.*, "A Fundamental Theorem...", pp. 400-401; 407-408] the problem of disaggregation concerns the art of making inferences about $(I - A)^{-1}$, where A is the *ideal* structural matrix, from the knowledge of A^* and $(I - A^*)^{-1}$, where A^* is the aggregated structural matrix that is empirically available. He sets out a method to approximate the *ideal* coefficient matrix A and its Leontief inverse $(I - A)^{-1}$ by augmenting the aggregated coefficient matrix A^* and its Leontief inverse $(I - A^*)^{-1}$, respectively, and then to estimate the *true* Leontief inverse $(I - A)^{-1}$ from this approximation.

[80] Fisher, *loc. cit.* Rubin [see Rubin *loc cit.]*, who has conducted tests on the various aggregation criteria, has concluded that the minimum distance criterion provides a theoretically valid and practical aggregation procedure for a variety of problems.

[81] McCarthy, *loc. cit.* The rank criterion has received limited testing by McCarthy.

[82] Skolka, *loc. cit.* The information loss criterion has received limited testing by Skolka.

[83] Henri Theil, *Economics and Information Theory* (Amsterdam: North-Holland Publishing Co., 1967), pp. 331-354. Also see Theil and Uribe, *loc. cit.*

has been paid to the empirical nature of the *aggregation bias.* The development of acceptable aggregation criteria is certainly quite important. It is also important, however, to find out more about the *aggregation bias* itself, the very existence of which has led to the development of the various aggregation criteria that are now available.

Of particular interest here, as far as this dissertation is concerned, is to find out to what extent input-output prediction errors that are directly attributable to aggregation (i.e., through the *aggregation bias*) are affected by alternative empirical specifications of the intersectoral flows matrix (and the *technological* coefficients matrix), as well as by alternative formulations of the model with respect to the treatment of competitive imports. An investigation of this problem is important for the simple reason that the application of *optimal* aggregation criteria to an existing input-output system that is structurally deficient will only lead to deficient results.

An existing input-output system may be called *structurally deficient* if, because of a particular set of empirical procedures used in the measurement of intersectoral flows (and the *technological* coefficients matrix) and because of the model's formulation with respect to the treatment of competitive imports, the aggregation component of input-output prediction errors is significantly greater than an alternative system with a different structure. Such a structural contrast is illustrated, on the one hand, by a system in which intersectoral flows are measured, if possible, as flows of products into *industries*, with each *industry* producing one or more goods, and, on the other hand, by a system in which intersectoral flows are measured in a theoretically more acceptable manner as flows of homogeneous products into *processes,* each producing only one, homogeneous good. The contrast can perhaps be made clearer when described in terms of the implied production functions (or *input* functions, in a more restricted sense): while one system contains, on the whole, *industry* Leontief production functions, the other contains *product* Leontief production functions. Unless an *industry* produces a single *homogeneous* good, the production functions (i.e., the input vectors) for the same sector would quite obviously show empirical differences under the two systems. Finally, to extend the contrast even further, either of these two systems may be designed in such a way that competitive imports are assumed to be either endogenous or exogenous in predictive applications.

In short, which model form leads to the least prediction error component due to aggregation alone is a fundamental problem that has not been carefully examined in the past and is investigated in this dissertation through the *experiments on the aggregation problem* reported in Chapter IV. The results should enhance our understanding of the comparative theoretical, as well as empirical, advantages and disadvantages of alternative input-output model formulations in economic analysis.

In keeping with the particular interest in this dissertation on the aggregation problem as just expressed, it is important to present here a mathematical exposition of the *aggregation bias,* which is central to the entire discussion. Similar expositions can be found in Green,[84] Theil,[85] and Ginsburgh.[86]

As a starting point, let us assume that we have an input-output model with n sectors and that we want to aggregate the model into m sets of sectors $I_1, I_2, \ldots, I_m$ in such a way that each sector belongs to exactly one set or *group.* We can introduce the mxn summation matrix

(1.32)
$$E = \begin{array}{c} \begin{array}{cccc} \overbrace{\quad\quad\quad}^{I_1} & \overbrace{\quad\quad\quad}^{I_2} & & \overbrace{\quad\quad\quad}^{I_m} \end{array} \\ \begin{bmatrix} 1 \ldots 1 & 0 \ldots 0 & \ldots & 0 \ldots 0 \\ 0 \ldots 0 & 1 \ldots 1 & \ldots & 0 \ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 \ldots 0 & 0 \ldots 0 & \ldots & 1 \ldots 1 \end{bmatrix} \end{array}$$

where each column corresponds to a sector and each row to a group. The sectors (i.e., columns) are so arranged that the first set of columns belongs to the first group, as indicated by 1 ... 1 in the first row and zeroes in all subsequent rows, the second set of columns belongs to the second group as indicated by 1 ... 1 in the second row and zeroes in all other rows, and so on. It is easily seen that

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[84] H. A. John Green, *Aggregation in Economic Analysis: An Introductory Survey* (Princeton, N. J.: Princeton University Press, 1964), pp. 69–78.

[85] Theil, *op. cit.* , pp. 322–331, *[Economics and Information* Theory].

[86] Ginsburgh, *loc. cit.*

$$EX = \begin{pmatrix} \sum_{i \in I_1} x_i \\ \cdot \\ \cdot \\ \cdot \\ \sum_{i \in I_m} x_i \end{pmatrix}, \quad EY = \begin{pmatrix} \sum_{i \in I_1} y_i \\ \cdot \\ \cdot \\ \cdot \\ \sum_{i \in I_m} y_i \end{pmatrix} \tag{1.33}$$

are the vectors of total output and of final demand, respectively, by sets of sectors.

Next, we consider the *technological* coefficients matrix after aggregation. Take the flow from I_1 to I_2, which is clearly the double sum of x_{ij} over $i\epsilon I_1$, $j\epsilon I_2$. Dividing this by the total output of the buying industry set I_2 (which is the sum of x_j over $j\epsilon I_2$), we obtain the input or *technological* coefficient. This coefficient is

$$\frac{\sum_{i\epsilon I_1} \sum_{j\epsilon I_2} x_{ij}}{\sum_{j\epsilon I_2} x_j} = \sum_{i\epsilon I_1} \sum_{j\epsilon I_2} a_{ij} \frac{x_j}{\sum_{j\epsilon I_2}} = \sum_{i\epsilon I_1} \sum_{j\epsilon I_2} a_{ij} v_j \tag{1.34}$$

where v_j is the share of the jth industry in the total output of its set I_2. We can now introduce the mxn matrix of *weights* V,

$$V = \begin{bmatrix} v_{11} \cdots v_{1I_1} & 0 \ \ldots 0 & \ldots & 0 \ \ldots 0 \\ 0 \ \ldots 0 & v_{21} \cdots v_{2I_2} & \ldots & 0 \ \ldots 0 \\ \cdot\cdot\cdot\cdot\cdot\cdot & \cdot\cdot\cdot\cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot\cdot\cdot\cdot \\ 0 \ \ldots 0 & 0 \ldots 0 & \ldots & v_{m1} \cdots v_{mI_m} \end{bmatrix} \tag{1.35}$$

where

$$\sum_{j=1}^{I_1} v_{ij} = 1; \ \sum_{j=1}^{I_2} v_{ij} = 1; \ldots; \ \sum_{j=1}^{I_m} v_{ij} = 1. \tag{1.36}$$

It can be seen that V is of precisely the same form as E except that the unit elements in E are replaced by the output shares of the individual industries in their sets. The relationship between E and V can be easily established as we see that

$$E\,V' = 1 \qquad \text{(of order nxn).} \tag{1.37}$$

We can now apply these results to the aggregation problem. We have, to begin with, a detailed matrix A, of order nxn. After aggregation, A is replaced by the matrix $E\,A\,V'$, where $E\,A\,V'$ is the mxm aggregated matrix of *technological* coefficients. Similarly, the vectors $X = (x_i)$ and $Y = (y_i)$ are replaced by EX and EY, respectively. Using the aggregated *technological* coefficients matrix and the aggregated final demand vector, we can obtain predictions of aggregated output levels as follows:

$$EX = (I - EAV')^{-1}\,EY. \tag{1.38}$$

We know, at the same time, that EX is actually equal to

$$EX = E(I - A)^{-1}\,Y. \tag{1.39}$$

We subtract (1.39) from (1.38) and conclude that

$$(I - EAV')^{-1}\;EY - E\,(I - A)^{-1}\;Y = GY \tag{1.40}$$

with G defined as

$$G = (I - EAV')^{-1}\;E - E\,(I - A)^{-1}. \tag{1.41}$$

Here, G is the mxn error matrix committed by the aggregation procedure. The mxl error vector GY, where Y is the final demand vector for the base year, is called the *aggregation bias*. Through a series of algebraic manipulations, the crucial matrix G can be expressed in a number of alternative ways, as has been done by Theil.[87] If Y were to refer to a prediction year (e.g., $Y_{t+\tau}$), then $GY_{t+\tau}$ would indicate the *aggregation bias* component of the prediction error vector.

$$EX_{t+\tau} - EX^p_{t+\tau} = E\,\Delta X^p_{t+\tau} \tag{1.42}$$

where $X_{t+\tau}$ and $X^p_{t+\tau}$ represent, respectively, the *actual* and *predicted* output vectors, $\Delta X^p_{t+\tau}$ is the error vector, and E is as explained before. Due to the *aggregation bias,* the prediction vector would be bounded by $GY_{t+\tau}$ as follows:

$$EX^p_{t+\tau} - GY_{t+\tau} \leqslant EX^p_{t+\tau} \leqslant EX^p_{t+\tau} + GY_{t+\tau}. \tag{1.43}$$

G. CONCLUDING REMARKS

The introductory discussion given in this chapter has been kept as simple as possible in order to maintain clarity in exposition. Throughout the chapter, it has been assumed that there exists a one-to-one correspondence between products and industries, implying that we have *commodity* or *product* production functions, as originally formulated by Leontief. Further, competitive and noncompetitive imports have been assumed away, implying that we have a semi-closed economy, exporting goods and services but not receiving any imports.

In real life, however, neither of these two simplifying assumptions are tenable. The basic information used in input-output model construction is rarely, if ever, collected in terms of *product-to-product flows*. In most instances, imports cannot be assumed away and must be explicitly recognized. In fact, the particular methods used in the treatment of imports lead to alternative model formulations, as shown in Chapter V.

In a general sense, this dissertation represents an exploration of the complications that arise when one removes these two simplifying assumptions, the methods that can be used to deal with these complications, the empirical consequences of these respective methods, and the alternatives that are available in input-output model formulation for national, regional, and multiregional economic analysis.

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[87] Theil, *op. cit.*, pp. 325–326, *[Economics and Information Theory]*.

CHAPTER II

MEASUREMENT PROBLEMS IN INPUT-OUTPUT MODEL CONSTRUCTION: A CRITICAL REVIEW AND SUGGESTIONS

A. INTRODUCTION

The conceptual and empirical problems faced in input-output model construction and the methods used to overcome them comprise the *hidden dimensions* of input-output analysis that are generally little understood or appreciated. The term *hidden dimensions* should not be used too lightly. The fact that a series of fundamental measurement problems and the methods used to overcome them have remained as *hidden dimensions* of input-output analysis over so many years has not been entirely accidental. Generally, these *hidden dimensions* have been relegated to appendices or footnotes, where they are rarely explained fully or clearly or even with candor. The situation presents considerable irony from an academic viewpoint, in that the less visible these *hidden dimensions* are the less they are questioned in free academic debate and the more the faults go unnoticed that violate in one way or another even the most modest standards of scientific measurement. Thus, the discussion given in this chapter draws attention to these *hidden dimensions* and makes them fully visible by critically studying them.

At the same time, this chapter suggests a few important reforms in input-output model construction, particularly in the numerical specification of the *technological* coefficients matrices, that should be widely adopted immediately. The properties of the suggested approaches are studied in detail through the series of experiments reported in Chapter IV.

The measurement problems discussed in the earlier part of this chapter are both conceptual and empirical, the distinction between the two being to a large extent arbitrary. It should perhaps suffice, for the present purposes, to note that while *conceptual* problems may refer to problems inherent in setting up rules or procedures that are in agreement with known or established theory, *empirical* problems may refer to the actual practical implementation of the established rules or procedures by following particular methods in light of existing data and other constraints. Ultimately, conceptual problems are resolved on very practical grounds, thus making the distinction not too meaningful for further discussion.

All of the conceptual and empirical problems faced in input-output analysis and means, procedures, methods used to overcome them are impossible to cover here. Thus, the emphasis in this chapter will be on the major issues that are of long standing in empirical input-output research. Data problems are completely omitted, for quite obvious reasons, but measurement problems arising from the organization of the available data in a particular way (e.g., availability of the basic intersectoral flows data in *product-to-industry flows* terms, etc.) are thoroughly examined. Further, specific research steps that might be followed in empirical input-output model construction are left out. Finally, measurement problems pertaining to particular industries or industry groups, such as the service sectors,are not included.

The plan of this chapter is as follows. First, some conceptual issues pertaining to the coverage of intersectoral flows in the economy are examined. This is followed by a discussion of statistical units adopted in the compilation of the basic data used in input-output analysis and the difficulties encountered in sectoral classification. Next, methods used in the valuation of intersectoral flows and the treatment of the distributive service or *margin* industries (e.g., transportation, retail and wholesale trade, insurance) are analyzed. Subsequently, a critical investigation is made of the treatment of secondary products, joint products, unallocated flows, and of the creation of *dummy* sectors. This is followed by a discussion of the treatment of imports, which completes the coverage of major issues pertaining mostly to the measurement of intersectoral flows. Then, much of the earlier discussion is brought together in a critical assessment of the prevalent practices used, particularly in the United States, in the measurement or numerical specification of the *technological* coefficients matrices. Finally, two alternative approaches are suggested, based on *commodity technology* and *industry technology* assumptions, respectively, for the numerical measurement and use of the *technological* coefficients matrices.

Throughout the chapter, frequent references are made to a variety of issues arising from the 1947 and 1958 input-output studies of the United States. In fact, the reservoir of knowledge gained and the problems faced in these two studies are fully incorporated into the writing of this chapter. A series of serious criticisms are directed at various aspects of these two studies. In general, the aim here has not been one of looking for faults just for the sake of exposing them but one of seeking out numerous illustrations from these two studies to provide or establish conceptual clarity and empirical precision.

B. COVERAGE: SOME CONCEPTUAL ISSUES

In empirically constructing an input-output table for a country, region, or an urban area, only *real* economic flows of goods and services during a particular accounting period (usually a year) are considered.[1] Real economic flows refer to market transactions, subject to certain modifications. Money flows or financial transactions, including loans, advances, security purchases and sales, transfers of ownership of existing physical assets not produced in the current (base) year or charged against current output (e.g., sales and purchases of existing improved and unimproved real estate, second hand machinery, used cars, airplanes, vessels, etc.) are excluded. On the other hand, the trade margins placed by the trade sectors or the freight provided by the transportation sectors in marketing used items are generally recorded, as in the 1947 Interindustry Relations Study of the United States, as real transactions. Adhering to this principle results in circumscribing or even eliminating many foreign trade transactions that are normally thought pertinent. For example, long and short term capital flows and changes in gold stock are completely omitted.[2]

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[1]Philip M. Ritz and Gabriel G. Rudney, *The 1947 Interindustry Relations Study, Industry Reports, General Explanations,* U.S. Department of Labor, Bureau of Labor Statistics, Division of Interindustry Economics, BLS Report No. 9 (March, 1953), p. 3.

[2]Murray Weitzman and Philip M. Ritz, "Foreign Trade", in Philip M. Ritz (ed.), *Input-output Analysis, Technical Supplement,* Conference on Research in Income and Wealth (New York: National Bureau of Economic Research, Inc., 1954), Part 3, p. 5. This supplement to *Input-Output Analysis: An Appraisal,* Studies in Income and Wealth, Vol. 18, contains eleven papers devoted to the description of the studies underlying the 1947 Interindustry Relations Study and was prepared by members of the U.S. Department of Labor, Bureau of Labor Statistics, Division of Interindustry Economics. Technically, it is perhaps the most exhaustive document dealing with empirical problems and solutions in input-output analysis published anywhere and still serves as a classic reference source to researchers in the field.

It should be noted that just as all monetary transactions are not included, so all non-monetary transactions are not excluded. For example, the value of animal fodder grown on farms and consumed by farm stock would be included.[3] Likewise, the value of food and fuel produced and consumed on farms, or the value of coal consumed by miners would be included, if they could be properly inputed.[4] Thus, in principle, barter transactions in a peasant economy would be counted. Discounting the remote possibility that flows would be measured in their natural or physical units, all entries would be shown in terms of monetary units, such as thousands or millions of dollars, guilders, etc., based on average prices prevailing during the accounting period.

The term *flows* can be given a more concrete meaning by making two points. First, delivery dates for goods may differ from consumption dates in the purchasing industries (or final demand sectors, such as government) as a result of stock changes in the purchasing industries. Thus, if entries on interindustry transactions in an input-output table were to be made on the basis of deliveries during a given year, then, clearly, a given input column would not always show the value of the respective inputs actually consumed by the purchasing industry. The direct implication of this, of course, is that a column of input coefficients describing the *technology* of an industry might be strongly influenced by additions to or withdrawals from stocks of raw materials, etc. This would obviously diminish the utility of the resulting input-output table for analytical purposes. Thus, input-output tables are actually more useful when they are drawn up in terms of consumption data rather than in terms of sales or purchases data. Then, each input vector would show the values (quantities) of goods and services that each industry has actually used up in the course of production. If the table is based on consumption data, then all changes in stocks (i.e., inventories) are entered in a final demand column entitled *net inventory change* or *increase of stocks.*[5]

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[3]United Nations, *Problems of Input-Output Tables and Analysis,* ST/STAT/SER.F/14 (New York: United Nations, 1966), p. 37.

[4]See, for example, U.S., Department of Commerce, Office of Business Economics, National Income Division, *The 1958 Interindustry Relations Study,* unpublished preliminary report, (November, 1964), Appendix 2.

[5]See, for example, The Netherlands Central Bureau of Statistics, *Input-Output Tables for the Netherlands,* Statistical Studies, No. 16 (Zeist, Netherlands: Uitgeversmaatschappij W. de Haan N.V., July, 1963), p. 8.

The second comment that should be made in relation to *flows* is that generally a distinction is made between *structurally related* and *autonomous* flows, where the former refer to flows for intermediate consumption and the latter refer to flows for final consumption.[6] All *structurally related* flows represent transactions on *current account* only (i.e., for current consumption). Thus, purchases by an industry of capital goods, that is its purchases on *capital account* (i.e., contributing to capital accumulation) are not shown as inputs to that industry. All purchases on *capital account* from each producing industry are combined and shown as one entry in the *gross private domestic investment* column, which is one of the final demand sectors. Thus, an input-output table records only each industry's *total* deliveries of capital goods to other industries; industry of destination is not shown.[7] The distinction between flows on *current account* and *capital account* is fairly arbitrary, depending on the accounting methods and conventions used in a particular country. In the 1947 Interindustry Relations Study of the United States, for example, the principle of a three-year life was used as a device for identifying capital equipment.[8] By contrast, in the 1958 Input-Output Study, an average life of greater than one year was used for identifying flows on *capital* account.[9] This latter principle appears to be more generally accepted. Some of the problems faced in uniformly applying such a principle or guiding criterion can be illustrated by the fact that in the 1947 Interindustry Relations Study, if a capital item with a useful life greater than three years was charged by an industry to *current* account, rather than to *capital* account as it should have been, such an item was actually shown in the input-output table as a *structurally related* flow (i.e., as an intermediate consumption rather than as capital accumulation).[10]

In the exogenous *foreign trade* or *exports* column, the distinction between *current* and *capital* account flows becomes completely blurred, since usually no distinction is even attempted. Further, the distinction loses much of its clarity in the *personal consumption*

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[6] Sidney A. Jaffe, "Final Demand Sectors", in Philip M. Ritz (ed.), *Input-Output Analysis, Technical Supplement,* Conference on Research in Income and Wealth (New York: National Bureau of Economic Research, Inc., 1954), Part 1, pp. 4-5.

[7] For some of the distinctions made here, also refer to Ritz and Rudney, *op.cit.,* pp. 5-6.

[8] Jaffe, *op. cit.,* Part 1, p. 17.

[9] U. S. Department of Commerce, Office of Business Economics, National Income Division, *loc. cit.*

[10] Jaffee, *loc. cit.*

expenditures (PCE) and *government purchases of goods and services* columns. For example, in the 1947 Interindustry Relations Study for the United States, as well as in the less ambitious 1958 study, personal consumption expenditures on new dwelling units for self occupancy were not included in the PCE column but entered in the *gross private fixed capital formation* column. That is, in both studies, the output of the *new construction* industry (defined as the *value-put-in-place* of all private new construction, additions, and alterations) was allocated to *gross private fixed capital formation.*[11] This meant that the *structurally related* flows for the new construction industry (i.e., the *new construction* row in the interindustry transactions matrix) had zero entries. Maintenance and repair construction, on the other hand, was treated as an industry distributing its output to all other industries and to the final demand sectors entirely for current consumption (i.e., on current account).

The distinction between *current* and *capital* account transactions is made even less clear in the treatment of *construction* flows into the *government* sector. In the 1958 Input-Output Study, for example, *new construction* and *maintenance and repair construction* were both shown in the *government* final demand column, as if they were purchases on *current* account.[12] Some of these apparent inconsistencies are explained away by the fact that, in keeping with national income accounting principles, the autonomous flows other than the purchases by households, government, and foreign countries comprise investment or capital accumulation in the system.

Whether or not certain *flows* appear in an input-output table as *autonomous* or *structurally related* flows is to a large extent determined arbitrarily. The treatment of *government enterprises* is a case in point. *Government enterprises* comprise of public functions that derive more than half of their current operating costs by the sale of goods and services to the general public.[13] Thus, in the 1958 Input-Output Study of the United States, public maintenance

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[11] U.S. Department of Commerce, Office of Business Economics, National Economics Division Staff, "The Transactions Table of the 1958 Input-Output Study, Revised Direct and Total Requirements Data", *Survey of Current Business,* XLIX, 9 (September, 1965), 33-49, 56. Also see Jaffee, *op cit.,* Part 1, p. 19.

[12] Norman Frumkin, "Construction Activity in the 1958 Input-Output Study", *Survey of Current Business*, XLV, 5 (May, 1965), 15.

[13] *Ibid.*, 13, footnote 2.

and repair construction for sewer and water facilities and for highway toll roads were shown as *structurally related* flows, since government enterprises were included in the endogenous part of the table. By contrast, public maintenance and repair construction for military facilities and freeways were shown in the exogenous *government* sector as *autonomous* flows.[14]

Apart from the few conceptual issues indicated here in respect to the coverage and measurement of *flows*, there are countless other problems of a different and quite technical nature that are inevitably faced in input-output analysis. A full discussion of these empirical problems lies beyond the scope of this study. Some of them, however, will be pointed out, even in cursory fashion, as part of the more general discussion presented here.

C. STATISTICAL UNITS AND SECTORAL CLASSIFICATION

In an input-output study, the economy must be broken into n ($n \geqslant 2$) mutually exclusive *endogenous* sectors. How these sectors should be formed is the problem of *classification*. In terms of set theory, each sector is a non-null set. If we can define the processing or producing segment of the economy as the universe, mutually exclusive or non-intersecting sets must be established, such that the complement of the collection of these sets is zero. Each set consists of elementary *statistical units* which may be *any* of the following four types: (1) a *product* or *commodity* group, such as agricultural products or iron and steel products, (2) an *establishment* group, such as a farm, a mine, or a factory, (3) an *activity*, such as trade or construction, or (4) an *institution*, such as a government enterprise or agency.[15] As a rule, a set or sector does not contain a mixture of these statistical units. Many of the difficulties that arise in input-output analysis are deeply rooted in having to define sectors in terms of these basic statistical units and in having to cope with the measurement problems that are inevitably generated as a direct consequence. Because of their particular importance in input-output analysis, the first two statistical units should be defined at somewhat greater length.

In the 1963 Census of Manufactures in the United States, information was collected on the output of approximately 10,000 individual product items.[16] The term *product* as used in the census programs represents the finest level of detail for which product information is requested from all industries, and it is not necessarily synonymous with the term *product* as

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[14] *Ibid.*

[15] United Nations, *op. cit.*, p. 31.

[16] U.S., Bureau of Census, *1963 Census of Manufactures,* Vol. II, *Industry Statistics,* Part 1, *Major Groups 20-28* (Washington, D.C.: U.S. Government Printing Office, 1966), p. 8.

used in the marketing sense or as used in industrial trade journals. In some cases it may be much more detailed and in other cases it may be much more aggregative. For example, *pharmaceutical preparations* was distributed in the 1963 Census of Manufactures into nearly 100 items, whereas *automotive gasoline* was reported as a single item.[17] For systematic coverage and reporting, a 7-digit number is used to identify an individual product, a 5-digit code for the class of product, and a 4-digit code for the total primary products in an industry.

The census of manufactures in the United States, as in almost all other countries, is conducted on an *establishment* basis. That is, a company operating establishments at more than one location is required to submit a separate report for each location. Also a company engaged in distinctly different lines of manufacturing activity at one location is required to submit separate reports if the plant records permit such a separation and if the activities are substantial in size. Census information based on establishment reports, therefore, differs substantially from those prepared on a company basis (i.e., from consolidated reports which not only combine activities at different locations, thus eliminating interplant transfers, but also include the non-manufacturing activities of companies primarily engaged in manufacturing). Since 1947, all establishments employing one or more persons at any time during the census year have been required to submit reports. In the 1939 and earlier censuses, establishments with less than $5,000 value of products were excluded.[18] This change in the minimum size limit in 1947 does not, however, appreciably affect the historical comparability of the census figures except for data on a number of establishments for a few industries.[19]

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[17] U.S. Bureau of the Census, *1963 Census of Manufactures, Industry Statistics,* MC63(2) Series (Washington, D.C.: U.S. Government Printing Office, 1966), Appendix A, p. 48

[18] *Ibid.,* Introduction, p. iv.

[19] In a special study of manufacturing plants with no employees in 1958 conducted by the U.S. Bureau of the Census it was found that while there were over 50,000 manufacturing establishments in this category in that year, they accounted for only about one-fourth of one percent of the total value of shipments of all manufacturing. See U.S. Bureau of the Census, *op. cit.,* p. 8. [*supra,* footnote 16].

In 1963 each of the establishments covered in the census was classified in one of approximately 425 4-digit manufacturing industries in accordance with the industry definitions contained in the Standard Industrial Classification (SIC) system.[20] Under this system a classification, an industry is generally defined as a group of establishments producing a single product or more or less closely related group of products. That is, the predominant classification criterion used is the similarity of products. This may take the form of a simple technological similarity, or it may be a closer grouping on the basis of homogeneity of function or appropriateness for use as a substitute or complement. Another basis for classification is the similarity of inputs. Thirdly, similarity of processes may also be used as a basis for classification. Finally, some industries may be simply residual categories having no clear basis for grouping. Clearly, boundary lines drawn according to these criteria may overlap.[21]

In practice, the boundaries of product-defined industries are established in such a way that, as a result of applying these criteria, each industry comprises a group of establishments whose output of *primary* products defining that industry account for a relatively high proportion of the products *primary* to it no matter where produced.[22]

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[20] See, for example, U. S. Technical Committee on Industrial Classification, *Standard Industrial Classification Manual 1967,* Executive Office of the President/Bureau of the Budget, prepared by the Office of Statistical Standards (Washington, D. C.: U. S. Government Printing Office, 1967).

This manual, which provides a classification structure for the entire national economy, was first issued in 1939. The following historical note contained in a recent publication by the U. S. Bureau of the Census [see *supra,* footnote 16, p.6], should be of general interest:

> For the manufacturing industries, a revised manual was issued in 1945, which, with minor modifications, was used in the 1947 Census of Manufactures. For the 1954 census, the classification structure used in 1947 was again employed, again with minor modifications. In 1957, the SIC system was extensively revised for manufacturing industries and historical comparability of some data was seriously affected. This revision and its effects on census series are described in the introduction and appendices to the 1958 census volumes. A minor revision of the SIC between 1958 and 1963 introduced some 4—digit industry changes but none of them crossed 3—digit group lines. Also, there were some amendments in industry titles, definitions, and index items, and additions to the index items in the SIC Manual.

An analysis of the historical comparability of industrial statistics is given in Harold T. Goldstein, *Historical Comparability of Census of Manufactures Industries, 1929—1958,* U. S. Bureau of the Census Working Paper No. 9 (Washington, D. C.: U. S. Government Printing Office, 1959).

[21] For a discussion of these points, see James W. McKie, *Industry Classification and Sector Measures of Industrial Production,* U.S. Bureau of the Census Working Paper No. 20 (Washington, D.C.: U.S. Government Printing Office, 1965), pp. 2-3.

[22] U.S., Bureau of the Census, *op.cit.,* p.6 [*supra,* footnote 16].

On the basis of the classification criteria just described, 425 manufacturing industries were established in the 1963 census. The SIC system operates in such a way that at one extreme are the 21 very broadly defined 2-digit manufacturing industry groups, and at the other about 10,000 individual 7-digit products. In between are spaced approximately 150 3-digit groups, 425 4-digit industries, and 1,100 5-digit product classes. The first four digits of a 7-digit product or a 5-digit product class indicate the industry which is primarily engaged in the production of that particular product or product class. Accordingly, an establishment is classified in a particular 4-digit industry if its production of the *primary* products of that industry exceeds in value its production of secondary products that are *primary* to other industries. Consequently, while some establishments produce only the *primary products* of the 4-digit industry in which they are classified, it rarely happens that all establishments in an industry specialize to that extent. This situation leads to the problem of *secondary products* in input-output analysis, which will be explained later in considerable detail.

In the 1958 Input-Output Study of the United States, two agricultural sectors and the *forestry and fishery products* sector were defined on a product or commodity basis, while six mining and fifty-two manufacturing sectors were defined on an establishment basis. On the other hand, the *agricultural, forestry, and fishery services, new construction, maintenance and repair construction,* and *wholesale and retail trade* sectors were defined on an activity basis. The same definition was followed, sometimes in a somewhat modified fashion, in the remaining, mostly service sectors. Of the total number of eighty-six sectors used in this study, three were *dummy* industries, four were set up solely for accounting purposes, and two represented government enterprises. In this study, an important criterion used in combining different products or activities into a single sector was the homogeneity or similarity of their input or use patterns. In a few cases, combinations were made when the output of one industry was used entirely by another industry. Lastly, small or miscellaneous type industries were frequently combined.[23]

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[23]U.S., Department of Commerce, Office of Business Economics, National Income Division, *op. cit.,* p.9 [*supra,* footnote 4].

It must be noted that while sectors as *consumers* are generally defined on the basis of the statistical units mentioned above, the sectors as *producers* are defined in *product* terms. Thus, a given *structurally related* flow, x_{ij}, represents the flow of *primary* product of type i to *product group, industry,* or *activity* j. As an example, we can think of the flow of livestock and the livestock products to the food and kindred products industry. This flow represents the total actual sales of livestock and livestock products, wherever produced, to the food and kindred products *industry.* This means that, since the industry in question also produces some secondary products, not all of the livestock and livestock products consumed by it are embodied in the production of the *primary* products of the establishments comprising it. Hence, the resulting *direct input* or *technological* coefficient, $a_{ij} = \frac{x_{ij}}{x_j}$, where x_{ij} represents the total value of product i consumed by industry j, and x_j represents the total output of industry j (i.e., the value of its primary and secondary products, subject to a few modifications that will be explained later), is subject to *error* in the sense that it does *not* accurately represent the real quantity (value) of commodity i that is required to produce one unit (i.e., a dollar's worth) of *product* j (i.e., primary product group j, regardless of the industry producing it). It is certainly far more convenient to collect information on the inputs of materials or goods absorbed by an *establishment,* since it forms a conventional activity unit for accounting purposes, than on the inputs of specific *activities* within each establishment. Thus, although ideally information should be developed to discover the commodity input structure of each commodity or commodity class, this is seldom, if ever, attained, simply because the industrial statistics gathered in the censuses of production in most countries take the establishment as the statistical unit.

In principle, intersectoral flows can be traced on an *industry-to-industry* flows basis, rather than on the ideal *commodity-to-commodity* basis or on the conventional *commodity-to-industry* flows basis. This, however, would create its own share of conceptual and empirical difficulties. The most important of these problems are explained below as part of the discussion on the treatment of secondary products.

D. VALUATION OF INTERSECTORAL TRANSACTIONS AND THE TREATMENT OF THE DISTRIBUTIVE SERVICE INDUSTRIES

Any transaction or *flow* in an input-output table, expressed in monetary rather than in physical units, may be recorded in three different ways: (1) in terms of the price paid by the purchaser (i.e., in purchaser's price), (2) in terms of the price received by the producer (i.e., in producer's value or price), or (3) in terms of producer's price, inclusive of excise and other taxes levied on the producer. The difference between the purchaser's and the producer's price (inclusive of excise taxes) is equal to the cost of distribution, consisting of insurance, rail, water, truck, air, or pipeline transportation costs, warehousing and storage charges, and wholesale and retail margins.[24] While the producer's price (exclusive of taxes) corresponds to the price f.o.b. factory and the producer's price (inclusive of taxes) corresponds to the effective price *off-factory*, the purchaser's price represents the market price.

In the main, the total output of the *margin* or distributive industries is defined as the total *margin* added to intersectoral flows (excluding indirect taxes) in the process of distributing these flows from producers to consumers. The total output of the wholesale and retail sector, for example, is thus equal to the total volume of trade margins in the economy. In trade accounting terms, this is the difference between net sales and net costs of goods sold, with the latter usually excluding the cost of the goods purchased for resale. In economic terms, the trade output is the part of the user price added by the marketing function. In this sense, it reflects the *value added* to product in the process of making it available to an intermediate or final consumer.[25] Of course, as in the case of the transportation industry, the outputs of some distributive industries do not consist entirely of the *margin* services rendered but also include the value of other or *nonmargin* services directly sold to all sectors, such as the carrying of passengers and mail by the transportation industries. Consistent with these definitions of output for the distributive industries, their inputs are restricted to commodities and services used only in the operation of

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[24] W. Duane Evans and Marvin Hoffenberg, "The Interindustry Relations Study for 1947", *The Review of Economics and Statistics,* XXXIV, 2 (May, 1952),103.

[25] William I. Karr, "Trade", in Philip M. Ritz (ed.), *Input-Output Analysis, Technical Supplement,* Conference on Research in Income and Wealth (New York: National Bureau of Economic Research, Inc ., 1954), Part 10, pp. 4-5.

their basic productive function (e.g., gasoline purchased by the trucking industry for use in its own vehicles).[26]

If, for example, the trade sector were not bypassed in the manner just described, by treating it as if it were a processing sector whose services are purchased, then it would mean that all commodities would flow into a *black box* labelled *trade* and then be charged out to consumers in some aggregate form. This procedure would completely eliminate the direct economic linkages between producers and consumers, which constitutes an important purpose of making input-output tabulations in the first place, and would substitute instead a heterogeneous trading structure.[27]

The adoption of any of the three systems of recording input-output transactions seems to vary from one country to another, depending to a large extent on data availability and/or accounting preferences. In the 1947 U.S. Interindustry Relations Study, all flows were expressed in producer's value terms; all excise taxes on the goods and services purchased by an industry were added and charged to the purchaser in the aggregate, by type of excise.[28] By contrast, in the 1958 U.S. Input-Output Study, flows expressed in producer's prices included federal, state, and local excise taxes collected and paid by the producer.[29]

An excellent discussion of the alternative methods used in different countries, together with a detailed explanation of the solutions adopted in the French input-output model for 1956, has been given by Dappe and Delange.[30] In general, it seems desirable to have the transportation, insurance, and trade margins associated with each flow recorded separately, in the form of additional *margin* matrices. If this is not possible due to data limitations, then the producer's value method, either inclusive or exclusive of taxes, should be preferred over the purchaser's value method, for a number of reasons. Essentially, using purchaser's

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[26] Philip M. Ritz and Gabriel G. Rudney, *General Explanations of the 200 Sector Tables: The 1947 Interindustry Relations Study,* U.S. Department of Commerce, Bureau of Labor Statistics, BLS Report No. 33 (June, 1953), p. 28.

[27] Evans and Hoffenberg, *loc.cit.*

[28] Jaffe, *op.cit.,* Part 1, p. 36.

[29] U.S., Department of Commerce, Office of Business Economics, National Income Division, *op.cit.* p. 8, footnote 1.

[30] Dappe and Delange (*sic*), "Traitement des Commerces et des Transports dans les Tableaux d' échanges interindustriels", *Etudes de Comptabilité nationale,* No. 3 (1962), 6-33.

prices results in counting the cost of distribution associated with each flow *twice*, once as part of a sector's output distribution and again as part of a consuming sector's input.[31] If we suppose that the table is based on a *product-to-industry* flows system, then, clearly each *technological* coefficient, a_{ij}, will be subject to fluctuations due solely to shifts in the cost of distribution of each transaction, x_{ij}, or due to arbitrary changes in tax regulations.[32]

E. TREATMENT OF SECONDARY PRODUCTS

At the outset, a distinction must be made between *secondary products* and *joint products.* It has already been mentioned that an establishment taken as a statistical unit produces a *primary* product or product group, in addition to which it produces a range of *secondary* products. Thus, when establishments are grouped into an industry on the basis of their *primary* products, then the material inputs into that industry are used up not only to produce the industry's primary but also its secondary products. Strictly speaking, those secondary products of an industry whose production are technologically independent of the production of the industry's primary products should be referred to as *secondary.* On the other hand, those secondary products whose production requires the same or single technological process used in producing the industry's primary products should be referred to as *joint products.*[33] A discussion of the treatment of the latter in input-output studies will be taken up separately.

There are several types of secondary products.[34] One class consists of those which are produced principally to supply certain inputs into establishments classified in the same industry. An example is electricity generated by a plant mostly for its own use. Another class consists of those which are produced in strict proportion with the establishment's primary products, such as the equipment installation services provided by manufacturers of specialized machinery (e.g., conveyor belts, cranes, etc.). A third class is represented by secondary products which no industry produces as its primary products and which a few or many industries produce in relatively small amounts. One well known example is miscellaneous stationery

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[31] United Nations, *op.cit.,* p. 38.

[32] *Ibid.*

[33] For these two definitions, see *ibid.,* p. 42.

[34] *Ibid.,* pp. 44-45.

and equipment used in offices. In general, methods dealing with the *secondary products problem* as indicated below are concerned with secondary products of the first two types. The last type is usually treated as in the case of certain types of joint products, to be explained later, by creating *dummy* industries.

At least theoretically, the products of any given sector under the input-output assumption should be homogeneous, should have a single input-structure, should not be produced in other sectors, and should be a perfect substitute in all uses, both intermediate and final. It is, therefore, important in input-output model construction to rule out production of a given product (or homogeneous product group) in more than one sector. Thus, when the original statistical information on intersectoral flows is available mostly in terms of *product-to-industry* flows, as is often the case, the problem requiring solution is to estimate the value of input-coefficients for product groups on the basis of this information (i.e., in effect to convert *product-to-industry* flows into *product-to-product* flows). This is appropriately called the *secondary products problem.*

Various approaches have been advanced to solve the secondary products problem. In short-hand fashion, they can be labelled (1) aggregation, (2) industry-to-industry flows, (3) redefinition, and (4) transfer methods. Among these, by far the most appealing is the redefinition approach since it is most consistently in line with the general idea behind the construction of an input-output model. The first two methods attempt to avoid the secondary products problem, but in the process create a few extra problems that render them generally unacceptable. The last one or the transfer approach, meanwhile, is nothing more than a simple accounting device for achieving sectoral commodity balances and in a fundamental sense represents an extremely poor choice in solving the secondary products problem. These methods can now be explained in somewhat greater detail.

The *aggregation* method consists of combining industries in such a way as to either eliminate secondary products altogether or to minimize their importance. In practice, however, this method leads to the undesirable prospect of having to define industrial sectors at too aggregate a level, which means that any given sector may contain a quite heterogeneous input pattern (i.e., multiple processes). There is some justification, therefore, in calling it an *out-of-the-frying-pan-into-the-fire* method.[35]

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[35] Wassily Leontief, *et al., Studies in the Structure of the American Economy* (New York: Oxford University Press, 1953), p. 500.

The *industry-to-industry flow* method keeps the *establishment* as the basic statistical unit, and defines both the producing and consuming sectors as *industries* (i.e., defines each industry as a group of establishments). While it avoids the secondary products problem, this method offers an intersectoral flows accounting system that would be unacceptable for a variety of reasons. Empirically, it would require the tracing of each commodity absorbed by an industry in terms of the *establishment* or group of establishments (i.e., industry) producing it. This in no way would be an easier task to perform than having to sort out the specific activities within each establishment in order to discover the commodity input structure of individual commodities produced by each specific activity (or process) in that establishment. Conceptually, the *homogeneity* and *process* assumptions of the input-output model would be violated, since each industry's output would be a mixed-bag of products, not necessarily homogeneous, and each industry would be employing multiple processes in the production of its primary and secondary products. There would be no compelling reason to suppose that each industry's secondary and primary output levels would increase in a constant proportion. Consequently, each industry's input-coefficients vector would be subject to instability due directly to variations in its output of secondary and primary products, which numerically act as weights in the determination of the industry's input coefficients. This built-in instability into the *technological* coefficients matrix would diminish its operational usefulness. Further operational problems will be created. For example, in using the model for prediction purposes, each element in the final demand vector (i.e., a product or product group) must be specified by the particular *industry* producing it. Each predicted output level would then numerically be a composite amount, including not only an industry's primary but also its secondary products. That is, it would not be possible to determine how much of each product or homogeneous product group would be produced by any single industry (or by all industries) except via the application of a separate allocation system designed to apportion an industry's total output into its primary products on the one hand and into every one of its (sometimes many) secondary products, on the other.

Under the *redefinition* approach, the values of the various goods and services used up in the production of an industry's secondary products would be subtracted from that industry's input vector and added to the input vector of the industry to which these secondary products are primary. In this way, each resulting input vector (sum of many input vectors) would represent the goods and services required to produce a homogeneous product group. In other words, a *product-to-product* flows basis would thus be approximated.[36] A mathematical formulation of the process underlying this approach is given by Edmonston.[37] A more compact and clearer formulation is given by this writer later in this chapter. The redefinition process can be empirically carried out by making the assumption, for example, that the input structure of a group of secondary products in any industry is the same as the input structure of the industry in which this group of secondary products is primarily produced.[38] This assumption may be abandoned in cases where the input structure of a group of secondary products in an industry shows a greater similarity to the input structure of the primary products of that industry, rather than to the input structure of the industry in which they are primarily produced.

In concept, the redefinition approach is most attractive, since the output of each sector would consist only of a homogeneous set of products, wherever produced. Moreover, the inputs of each sector would reflect the goods and services required in the production of a homogeneous set of products. In practice, however, the applicability of this approach is often hampered by the existence of inadequate data to support the complicated adjustments that are necessary. In addition, a criticism usually directed at this approach makes the contention that using or approximating a *product-to-product flows* system in input-output tabulations creates a wide divergence from the existing industrial structure defined in terms of establishments and, consequently, makes it difficult to translate the results of input-output analysis into actual industry terms (i.e., makes it difficult to trace the effects of exogenous changes on specific *industries* defined on an establishment basis). This disadvantage, however, can be easily overcome through the use of an *industry mix* matrix that will be discussed later in this chapter. As in the case of the *technological* coefficients matrix, an *industry mix*

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[36] United Nations, *op. cit.*, p. 42. Also see Bengt Höglund and Lars Werin, *The Production System of the Swedish Economy, An Input-Output Study* (Stockholm: Almqvist and Wiksell, 1964), pp. 48-52.

[37] J. Harvey Edmonston, "A Treatment of Multiple-Process Industries", *The Quarterly Journal of Economics,* LXVI, 4 (November, 1952), 557-571.

[38] It should be noted that such an assumption would be consistent with the definition of secondary products .

matrix can be appropriately adjusted for anticipated changes over time, for example by applying the RAS method.[39]

We can now turn to a discussion of the *transfer* method of handling secondary products. The most well-known version of this method is one in which a particular secondary product (product group) of a producing sector i that is primary to some other industry j (i.e., x^s_{ij}) is shown in the input-output table as a *fictitious* sale from the producing sector i to the latter, consuming industry j.[40] Suppose that the *real* transaction between sectors i and j, expressed in producer's prices, is noted as x_{ij}. Then, using the transfer device, this transaction is artificially increased by the amount x^s_{ij}. To put it another way, let us imagine a matrix, denoted as $X^s = [x^s_{ij}]$, in which every element, x^s_{ij}, shows simply the value of a secondary product (product group) actually produced by sector i that happens to be primarily produced by industry j. If we can also imagine an input-output table compiled in terms of *product-to-industry flows,* denoted as $X = [x_{ij}]$, then the transfer method requires that we add these two matrices, thus obtaining

$$X^L = X + X^s = [(x_{ij} + x^s_{ij})] \tag{2.1}$$

where the superscript "L" stands for *Leontief,* since this method of measuring intersectoral flows by handling secondary products in this particular way is historically associated with him.[41] For completeness, we should note that here, x_{ij} *may* include competitive imports (to be defined shortly) of type i used in the production of industry j's primary and secondary products.

Quite obviously, treating secondary products in this manner is empirically indefensible, since it knowingly distorts the facts in measuring the input requirements of each industry and, of course, in deriving the *technological* coefficients matrix. As will be seen later, this practice of misrepresentation extends into the definition of industry output levels in deriving the *technological* coefficients matrix, thus subjecting each coefficient to what amounts

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[39] For references already cited on this subject, see footnotes 28 and 65 in Chapter I.

[40] See, among others, United Nations, *op. cit.,* pp. 43-44; Evans and Hoffenberg, *op. cit.,* 105-106; and Höglund and Werin, *op. cit.,* pp. 56-58.

[41] Leontief, *et al., op. cit.,* p. 500.

to deliberate observation error by misrepresenting both the numerator and the denominator. Moreover, using the transfer method in dealing with secondary products results in double counting in the measurement of intersectoral flows. Suppose, for example, that the *household furniture* industry produces, as secondary products, five million dollars' worth of *glass and glass products*. This amount is not only shown as part of the total output of *glass and glass products*, distributed to all intermediate and final consumers, but is also shown under the transfer method as a *fictitious* sale from the *household furniture* sector to the *glass and glass product* industry.

The transfer method described here has at least two variants.[42] Under the first, the secondary products of sector i that are primary to industry j, x_{ij}^{s}, are shown as a positive *fictitious* sale from i to j, but this is offset by subtracting x_{ij}^{s} from industry j's intra-industry transactions (i.e., x_{jj}). Under the second, x_{ij}^{s} is still shown as a *fictitious* sale from sector i to sector j, but this is offset by subtracting x_{ij}^{s} from sector j's sales to sector i. In the final analysis, these variants amount to nothing more than additional accounting devices and offer no solution to the problem.

The only conceivable justification for the use of the transfer method is for a reason that has little to do with the accurate measurement of intersectoral flows: maintaining material balances in the input-output table (i.e., making total inputs of a sector equal its total output). How sectoral balances are achieved in an input-output table will be apparent in a discussion later in this chapter. It should suffice to emphasize here only that achieving sectoral balances is purely a mechanical process to insure and demonstrate accounting consistency in the input-output table. There is no reason why a table showing material balances could not be developed and presented as a separate, supplementary table. This would by no means solve the secondary products problem, but would certainly eliminate the obvious observation errors inherent in most input-output studies, including those compiled for the United States for 1947 and 1958.

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[42] United Nations, *op. cit.*, p. 43. In footnote 6, reference is made to Alan M. Strout, "Disaggregation of an Industry Production Function When it is Desired to Treat Individual Industry Joint Products in Separate Input-Output Table Rows" (Cambridge, Mass.: Harvard University, Harvard Economic Research Project, 24 October 1962, Mimeographed); and "A Flexible Input-Output Convention for Secondary Product Transactions" (Cambridge, Mass.: Harvard University, Harvard Economic Research Project, 14 January 1963, Mimeographed).

In conclusion, there has generally been a failure, particularly in the United States, in coming to grips with the secondary products problem, or more broadly, with the theoretical and empirical problem posed by the fact that the basic information available for input-output model construction is compiled on an establishment basis. Specific suggestions are made later in this chapter for certain solutions to this general problem. These solutions are incorporated into the formulation of alternative models that are used in the experiments reported in Chapter IV.

F. JOINT PRODUCTS, DUMMY SECTORS, AND THE TREATMENT OF UNALLOCATED FLOWS

1. Joint Products and *Dummy* Sectors

Some industrial processes produce a number of entirely different products through the same process (i.e., common input structure). A few of the most well known examples of joint products are the slaughtering industry producing both meat and hides, cattle farming industry producing both milk and animals for slaughtering, petroleum refinery products, the production of coke and gas, and various types of rolled steel products. In certain cases, the proportions in which the different products may be produced are approximately fixed, while in others they are variable and depend on demand conditions. In general, several classes of joint products can be identified. First, the products may be of relatively equal importance and as a rule not produced by other industries, such as the production of mutton and wool, or petroleum refinery products. Alternatively, one of the products may be of distinctly lesser importance. Further, many industries may be producing, in addition to their dominant or primary product, the same type of by-product, such as metal and nonmetal scrap and waste products. In all these cases, certain empirical problems are created in input-output analysis that are difficult to resolve satisfactorily. The redefinition approach is out of the question, since a common process is used in the production of these joint products. The transfer approach, on the other hand, leads to the depiction of *fictitious* intersectoral flows in the system where no such flows actually exist. The use of the transfer approach in respect to the treatment of scrap and waste products can be explained to illustrate this point.

Usually, a *dummy* sector is created to which every sector producing scrap and waste products *sells* such products. Likewise, all scrap and waste products are pooled together into a *scrap and waste products sector,* and its output is distributed along a row to all consuming industries. Thus, while the *scrap and waste products* column shows every sector's total production of scrap and waste products, the corresponding row shows every sector's total consumption of such products. In this manner, scrap and waste products are excluded from intersectoral transactions, but still included as part of each sector's total output.[43] The total output of scrap and waste products includes, in addition to those produced by the productive sectors of the economy, the net sales by the final demand sectors (e.g., households) of scrap, used, and secondhand goods, including structures.[44] These net sales are entered in the input-output table as negative purchases by the final demand sectors from the scrap and waste products row.[45] Total *supply* of scrap and waste products, then, consists of both domestic *production* and imported scrap and waste products, where the latter are entered in a table as purchases of the scrap and waste products industry (column) from the imports row. The distributive margins associated with the scrap and waste product shipments appearing in the *dummy* column sector (including the margins associated with the imports of such products), accruing to the *margin* industries, are shown as *paid* by the scrap and waste products industry.[46] Similarly, distributive margins associated with each industry's consumption of scrap and waste products are shown as accruing to the *margin* industries.

In general, the scrap and waste products row and column are dropped from the *technological* coefficients matrix, just prior to the matrix inversion process. The effect of this practice is to avoid any exogenous demand for scrap and waste products from directly or indirectly generating primary output from any industry in the economy. In the 1947 Interindustry Study of the United States, a different practice was followed. Just prior to the inversion process, both the scrap and waste products and by-products shipments of each producing sector were combined and shown as intra-industry sales.[47] The justification given for this procedure was that it not only avoided the problem of indirect effects but also made possible the interpretation of total by-product production requirements resulting from the

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[43] Ritz and Rudney, *op. cit.*, p. 30 [*supra,* footnote 26].

[44] U. S., Department of Commerce, Office of Business Economics, National Income Division *op. cit.*, Appendix 1, p. 25 [*supra,* footnote 4].

[45] U. S., Department of Commerce, Office of Business Economics, National Economics Division Staff, *op. cit.*, Table 1, p. 39 [*supra,* footnote 11].

[46] This practice is in general contrast to the usual accounting procedures used under the *transfer* method of treating secondary products, where there are no distributive margins associated with secondary products transfers.

[47] Ritz and Rudney, *op. cit.*, pp. 55-56 [*supra,* footnote 26].

application of a stipulated bill of goods. Once the required output levels would have been calculated, it would then be a simple matter to compute associated output of scrap and waste products and other by-products, by applying, for example, the base year ratio of these products to each producing industry's total output. Finally, a consistency check would be conducted between total amount of generated scrap and waste products (and other by-products) on the one hand and the total expected consumption of such products, on the other, where consumption levels could be computed by using the base year input coefficients for these products.[48]

The important point not recognized in the given justification just described is that as a result of artifically *padding* the diagonal entries in the *technological* coefficients matrix, the observation errors thus introduced into the diagonal input coefficients are spread into every single element in the Leontief inverse. Since, theoretically, the determinants of both the A-matrix and the Leontief matrix are now made subject to error through the diagonal elements, and since every element in the Leontief inverse represents a division of every element in the adjoint of the Leontief matrix (which is now perturbed) by the determinant of the Leontief matrix, then the resulting Leontief inverse matrix, is affected in every element and is subject to error. Consequently, the contention that the procedure used in the 1947 Interindustry Relations Study avoids the problem of indirect effects of scrap and waste products (and other by-products) is simply not correct mathematically.

In the 1958 Input-Output Study, exactly the same procedure as in the 1947 Study was used with respect to the treatment of scrap and waste products and by-products; that is, they were shown as inputs into the producing industry rather than to the consuming industry. The published 1958 *technological* coefficients (i.e., direct requirements) matrix, as well as the Leontief inverse (i.e., total or direct *and* indirect input requirements) matrix, contain therefore, the same mistake as was incorporated into the published 1947 matrices.[49]

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[48] *Ibid.*

[49] See Morris R. Goldman, Martin L. Marimont, and Beatrice N. Vaccara, "The Interindustry Structure of the United States: A Report on the 1958 Input-Output Study," *Survey of Current Business,* XLIV, 11 (November, 1964), Table 2, footnote 1.

2. Unallocated Flows

In the compilation of input-output tables, a situation frequently faced is that after the output of a homogeneous product (product group) has been distributed to all consuming industries and final demand sectors, a residual amount cannot be traced to any specific consuming sector. An *unallocated* or *undistributed* column is thus set up in which to record these unallocated flows. Likewise, after each industry's known purchases of goods and services have been accounted for, the discrepancy between the industry's known *total* cost of materials, supplies, etc., and the sum of its accountable costs is entered in an *unallocated* or *undistributed* row. The row total should be expected to equal the column sum. Since unallocated amounts represent untraceable intersectoral flows, the unallocated *sectors* are included in the endogenous part of an input-output table.

Although the unallocated *sectors* may serve a useful accounting purpose in the compilation and presentation of an input-output table, they are rather embarrassing in analytical applications, since no meaning can be attached to a row and column of unallocated input coefficients. It would be obviously out of the question to have a Leontief inverse matrix with an unallocated row and column.

Often, a number of ways are used to distribute the unallocated flows to avoid these embarrassments. For example, a mechanical procedure may be used, such as distributing the residual evenly to all the other cells, among non-zero cells, or in proportion to the values of the entries in the non-zero cells. As Evans has pointed out, these mechanical methods of distributing unallocated items are likely to do little or no good.[50] Alternatively, they may be distributed largely on a judgmental basis, advisedly drawing as much as possible upon the expert knowledge of industry specialists. Stone has outlined a statistical distribution method, originally due to Durbin, which features a variance matrix expressing the willingness of those concerned to alter entries in the initial table so as to produce a final table that is balanced by the complete absorption of unallocated items, and which enables the consequences of these subjective judgments to be worked out simultaneously.[51]

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[50] W. Duane Evans, "Input-Output Computations," in Tibor Barna (ed.), *The Structural Interdependence of the Economy,* The Report of an International Conference Held in Italy in 1954 (New York: John Wiley and Sons, Inc., 1955), pp. 53-102.

[51] Richard Stone, *Input-Output and National Accounts,* (Paris: Organization for European Economic Cooperation, 1961), pp. 160-163.

In the 1958 Study, unallocated flows were distributed mostly on a judgmental basis, probably with little consultation with industry specialists.[52] In any event, no publicly available documents exist explaining just exactly how the distribution process was accomplished. The only information available on the subject is given in a computer tape containing the 1958 Study data file, which provides considerably more detailed information on intersectoral flows than contained in the published tables. Here, the distribution of unallocated flows is recorded explicitly.[53]

G. TREATMENT OF IMPORTS

The manner in which imports from the *rest of the world* are treated in the empirical construction of an input-output table results in alternative analytical model formulations, each having its own peculiar set of both theoretical and practical advantages and disadvantages.Since a rather complete coverage of the broader analytical options available in the treatment of imports in input-output models will be given in the last chapter, the focus here will be on some of the empirical problems faced in the valuation, classification, and treatment of imports.

First, and as already indicated, imports are generally divided into two categories – *competitive* (substitutable or supplementary) and *noncompetitive* (nonsubstitutable or complementary). *Competitive* imports are defined as those commodities or services which are identical or similar in nature to those produced by a domestic (national or regional) industry. *Noncompetitive* imports, on the other hand, are defined to comprise those imported goods and services for which there are no similar or substitutable products or services produced domestically.[54]

The main criterion used in the 1947 Interindustry Relations Study to distinguish between competitive and noncompetitive imports was whether there existed a counterpart item produced domestically.[55] In the 1958 Input-Output Study, substitutability was

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[52] Based on conversations with staff members of the OBE (U.S. Department of Commerce, Office of Business Economics), who have worked on the 1958 Input-Output Study.

[53] Intersectoral flows as recorded under Code 11.

[54] Murray Weitzman and Philip M. Ritz, "Foreign Trade," in Philip M. Ritz (ed.), *Input-Output Analysis, Technical Supplement,* Conference on Research in Income and Wealth (New York: National Bureau of Economic Research, Inc., 1954), Part 3, p. 7.

[55] Jaffe, *op. cit.,* Part 1, p. 31.

determined on a judgmental basis, applying the guide that the imported good or service should be interchangeable with a domestically produced item without any changes in the technology of the consuming industry or the resultant product.[56] In the 1947 Interindustry Relations Study of the United States, for example, natural rubber was considered to be a competitive import since for the most purposes it could readily be substituted for domestically produced synthetic rubber. On the other hand, products such as cacao beans, green coffee, manila hemp, and jute burlap were classified as noncompetitive imports. In addition, noncompetitive imports included net private and government unilaterals,[57] personal consumption expenditures of American citizens in foreign countries, and payments (principally by the government and by the ocean transportation industry) for goods and services received in foreign countries.[58] In the same study, personal consumption expenditures of foreign visitors and employees of foreign governments in the United States were subtracted from total consumption expenditures of U.S. citizens abroad, and the net difference was shown as a noncompetitive import entry in the household (personal consumption expenditures) column.[59] While this procedure was consistent with national income accounts concepts, it was faulty in the sense that it led to a downward bias, however small, in the household column entries. In analytical applications of the model, such a procedure is generally not quite desirable, since it forces the expenditures of foreign visitors, etc., in a country to have no effect at all on that country's domestic economy, thus, introducing a downward sectoral bias in input-output calculations. This can be quite serious, depending, of course, on the extent to which the economy under consideration is a tourism *importing* economy, on balance.

Secondly, competitive imports are valued at domestic port value (i.e., *landed cost* or *landed value*), while noncompetitive imports are valued at foreign port value. To the foreign port value of an imported commodity are added the costs of international transportation

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[56] Goldman, Marimont, and Vaccara, *op. cit.,* 17, footnote 13.

[57] Obtained by subtracting unilaterals received (including reverse lend-lease) *from* total unilateral grants (including goods, freight expenses, and cash payments). See Irving H. Licht, "Government," in Philip M. Ritz (ed.), *Input-Output Analysis,* Technical Supplement, Conference on Research in Income and Wealth (New York: National Bureau of Economic Research, Inc., 1954), Part 2, p. 23.

[58] Weitzman and Ritz, *loc. cit.*

[59] Jaffe, *op. cit.,* Part 1, p. 9.

and insurance provided by domestic shippers and insurers and duty paid to the domestic government. These costs are equivalent to the total *margin* or distribution costs associated with domestically produced and consumed goods and services, as discussed earlier, and are appropriately allocated to the particular *margin* industries providing such services.[60]

Thirdly, in both the 1947 and 1958 input-output studies of the United States, noncompetitive imports were distributed to the consuming industries and final demand sectors from a single source (row). Competitive imports, on the other hand, were treated differently. In the 1947 Study, they were routed to the consuming industries and final demand sectors as part of the output of counterpart domestic products. This meant that any given intersectoral flow x_{ij} or a final demand entry y_i could consist of either domestically produced products, competitive imports, or a combination of the two ($x_{ij} = x_{ij}^D + x_{ij}^M$; $y_i = y_i^D + y_i^M$). In the 1958 Study, only the competitive imports that were consumed as intermediate goods in the economy (i.e., imports used for further processing) were treated in this manner, which meant that the final demand sectors were shown as having purchased all their import requirements from the noncompetitive imports row (Row 80A).[61] This procedure, however, does not seem to have been followed in the 1958 Study in a consistent manner. For instance, imported cars do not appear in the published tables as a purchase from the *imports* row (Row 80A), but are rather shown as a purchase from the motor vehicles industry. The stated reason for this is that import data on autos and parts used for the 1958 Study did not distinguish between assembled and unassembled cars.[62]

In both the 1947 and 1958 studies, adding the domestic port value of competitive imports to the output of counterpart domestic products for distribution to all consuming industries meant that if, say, $1,154 million worth of food and kindred products entered the economy in 1958 as competitive imports, the domestic food and kindred products industry was shown as making *fictitious* purchases from the competitive imports sector

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[60]Weitzman and Ritz, *loc. cit.* Also see Ritz and Rudney, *op. cit.*, p. 32 [*supra,* footnote 26].

[61]Nancy W. Simon, "Personal Consumption Expenditures in the 1958 Input-Output Study," *Survey of Current Business,* XLIV, 10 (October, 1965), 18. The imports shown as having been purchased by final demand sectors from Row 80A were in substantially the same form as they entered the economy, not requiring further processing.

[62]*Ibid.*, 9, footnote 5.

(Row 80B) in the amount of $1,154 million, plus services from the transportation, trade, and insurance sectors responsible for bringing the imports to the domestic port of entry. This practice, which has been earlier identified as the *transfer* method, is generally acceptable in a supplementary table showing sectoral commodity balances. However, retaining such *fictitious* transactions in the denominator in the derivation of the *technological* coefficients matrix, the Leontief matrix, and finally the Leontief inverse matrix introduces errors into these matrices which diminish their analytical utility. In the 1958 Study, for instance, both competitive imports transfers (Row 80B) and noncompetitive imports (Row 80A) were combined into one row (Row 80) before computing the *technological* coefficients matrix, and this row was actually included in both the Leontief and the inverse matrices.[63] No particular reason has been given for this procedure, and it is difficult to think of a good one to justify it.

Developing an input-output table is only the beginning of an analytical process, and not an end in itself. Thus, an effort should be made in the empirical formulation of input-output tables to have them respond effectively to analytical needs. A step would be made in this direction if, in the future, the following suggestions are adopted as operational guidelines.

First, the competitive imports transfers row should be kept out of both the Leontief matrix and the inverse matrix, as it contains *fictitious* entries. These arbitrary *conventions* to assure sectoral commodity balances in the input-output table, where basic intersectoral flows information underlying the table is largely available only on a *commodity-to-industry* flows basis (as opposed to *commodity-to-commodity* flows basis), should simply be abandoned in the empirical estimation of the *technological* coefficients matrix, the Leontief matrix, and the Leontief inverse matrix.[64] There would be no harm in retaining these *conventions* in showing sectoral balances in supplementary tables, but the entire input-output effort is harmed when they directly affect, in a deleterious way, the estimation of structural parameters.

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[63] U.S., Department of Commerce, Office of Business Economics, National Economics Division Staff, *op. cit.,* Tables 2 and 3.

[64] In this context, we should also eliminate the transportation, trade, and insurance services accompanying each competitive imports transfer which are charged as inputs from these *margin* sectors to each consuming sector to which the competitive imports are transferred. In other words, the entire bundle of *fictitious* entries must be eliminated prior to the computation of the structural matrices.

Secondly, it is unsatisfactory to treat noncompetitive imports in a single row. Inasmuch as it is possible, an attempt should be made to develop noncompetitive imports flows *and* input coefficients matrices. Such matrices can be developed, for example, by showing the noncompetitive import of a commodity as being distributed to all consuming sectors by a row sector that would be producing such a commodity if it were produced domestically.[65] These matrices should be shown either separately or as an integral part of the basic input-output table. If the latter course is adopted, both the noncompetitive import flows and coefficients should be clearly and separately indicated in each *cell* of the table. In this way, it would be possible to predict the noncompetitive import requirements of the economy by commodity. Further, it would be possible to trace the implications of certain shifts in final consumption patterns (e.g., from domestically produced goods to competitive or noncompetitive imports, etc.) on domestic output levels and competitive and noncompetitive import requirements.

In conclusion, each cell of an input-output flows table should preferably contain separate entries for the input of domestically produced materials, competitive imports, and noncompetitive imports. The type of clarity and detail suggested here can already be found in input-output models constructed for the European Economic Community (E.E.C.) and member countries.

H. VALUATION OF TOTAL SECTORAL OUTPUT AND INPUT LEVELS AND THE DERIVATION OF TECHNOLOGICAL COEFFICIENTS IN THE U.S. INPUT-OUTPUT MODELS: A SYNTHESIS

According to input-output theory, total input of each sector must equal its total output. If both the producing and consuming sectors are defined in an identical way (i.e., as a homogeneous product group), this means that each column sum in an input-output table must equal each corresponding row sum, provided, of course, that there are no imports in the system. In the absence of imports, total input of a consuming sector is equal to total cost of

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[65] Alternatively, it may be shown as being distributed to all consuming sectors by a sector which produces it as a principal product in the country of origin. See, for instance, A. Koutsoyiannis and A. Ganas, *Input-Output Table of the Greek Economy (Year 1960).* (Athens: Center of Planning and Economic Research, 1967), p. 6. It should be noted, parenthetically, that here, all imports (both competitive and noncompetitive) were treated in this manner.

materials, supplies, etc., *plus* value added. Total output is then simply equal to the row sum (i.e., total deliveries to all intermediate and final demand sectors, reflecting total production during the accounting period).

In reality, two basic complications arise. First, imports, of course, do exist and cannot be ignored. Secondly, each producing and consuming sector is generally not defined and measured in an identical way: the producing sector is defined as a group of homogeneous products, wherever produced (including competitive imports, which are classified for distribution as part of the very domestic products with which they compete), while the same sector as a consumer is defined in *industry* terms (i.e., a domestic industry producing a group of primary products, and at the same time producing a variety of secondary products). This plainly indicates that each column sum will not and need not equal the corresponding row sum. As we shall see, through various devices sectoral balances are *forced* on the system. This would not be so objectionable, except for the fact that the sectoral balances so achieved are then taken quite seriously, rather than as a tolerable side exercise demonstrating a semblance of numerical consistency in the input-output table.

In reality, more often than not the basic intersectoral flows information is available in *product-to-industry flows* terms, which means, simply, that consuming sectors are mostly defined in *industry* terms while the producing sectors are defined in *product* terms (i.e., primary products, regardless of where produced). The pair-wise producer-consumer identity or symmetry is maintained by the fact that while as a producer a given sector is called, for example, the food and kindred products *sector* (i.e., a collection of all such products, whether produced as primary products by the food and kindred products industry or as a secondary product by all other industries, plus competitive imports), as a consumer the same sector is called the food and kindred products *industry*.

Total input of a consuming industry is equal to its total output. This equality can be illustrated in considerably more detail in the following accounting framework:[66]

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[66]This accounting framework represents, in its entirety, an independent contribution. The components of the *output* side have been developed through a study of some of the worksheets used in the 1958 Input-Output Study of the United States.

Input	**Output**
Cost of domestic materials consumed, whether produced by the primary industry or by other industries as a secondary product, in purchaser's value	Total value of shipments, producer's value
	Primary products
Cost of domestic materials consumed, in producer's value	Secondary products
	Contract work performed
Transportation costs, trade margins, insurance charges,	Miscellaneous receipts
Cost of competitive imports consumed, domestic port value	Scrap and salable refuse
Competitive imports, foreign port value	Repair work
	Research and development
Ocean freight, insurance, duty, and other *margin* charges	Other[67]
Cost of noncompetitive imports, foreign port value	Electrical energy sales
Noncompetitive imports, domestic port value	
Ocean freight, insurance, duty, and other *margin* charges	
Value added	
Net inventory change, wherever held	Net inventory change, wherever held
Total Input	Total (Industry) Output

[67] Excludes receipts for installation work.

The basic point that is being made here is that the production function, or more strictly, the input function, specifying the quantitative relationship between each input *and* a given consuming industry's output should not be confused with an alternative input function (*ideal* from the standpoint of input-output theory) specifying the quantitative relationship between each input and the output level of a given homogeneous product group, regardless of where produced. This distinction is of fundamental importance, since the core of the input-output problem is to approximate input functions of the latter type under circumstances where mostly input functions of the former type can be estimated from the available data.

When the basic intersectoral flows information is available mostly in *product-to-industry flows* terms, as is often the case, every input must be expressed as a linear function of total *industry* output:

$$x_{ij} = f(x_j) = c_{ij} + a_{ij}\, x_j \tag{2.2}$$

where c_{ij} is the regression constant, assumed to be zero under input-output theory, and a_{ij} is the *technological* coefficient, indicating the marginal (average) propensity of industry j to consume products of type i per unit of its own output.

Since the regression constant c_{ij} is assumed to be zero, the linear equation expressing the relationship between a given input and the consuming industry's total output is forced through the origin. The derivation of each *technological* coefficient is thus simplified to the point of expressing it plainly as the ratio of a given input to the consuming industry's total output (i.e., $a_{ij} = x_{ij}/x_j$).

At this point, it is important to emphasize the fact that in most input-output studies, where the underlying intersectoral flows data are organized mostly in *product-to-industry flows* terms, the simple and correct method just described is not used in the derivation of the *technological* coefficients. In most studies, including the input-output studies in the United States, each *technological* coefficient is derived as the ratio of a modified *input* to a modified sectoral *output* level. As a result of these modifications, each input is made to include not only the actual materials consumed, as noted above, but also secondary products transfers-in. That is, the secondary products of a producing (row) industry, that are produced primarily by the consuming industry (in its role as a producer), are shown as a *fictitious* input into the consuming industry. Thus, potentially, each element in the input vector of a consuming industry may include such *fictitious* components. Similarly, the row definition of sectoral output level is used as the denominator in computing the *technological* coefficients, rather than the *industry* output level, as noted above. The row definition of sectoral output includes not only the primary product shipments of a domestic industry, but also its secondary products (which are shown as *fictitious* inputs to industries along the row which primarily produce such products), the secondary products of other industries that are primarily produced by the row industry in question, and competitive imports (domestic port value) that compete with the primary products of the row industry in question. It can be seen immediately that the difference between *industry* output level *and* sectoral output level defined as the row sum is made up of two items: (a) secondary products transfers-in, and (b) competitive imports transfers-in. It can be seen, further, that these two items are shown as *fictitious* inputs into each consuming industry, so that ultimately the equality of each column sum (total input) and each row sum (total *commodity* output in the input-output table) is guaranteed. How this balance is achieved can be seen in the following accounting framework, in which total input is now defined to include secondary products transfers-in and competitive imports transfers-in, while total output is defined as the row-sum:[68]

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[68] This accounting framework represents a synthesis of the basic thinking involved in the 1958 Input-Output Study of the United States with respect to the achievement of sectoral balances in the published interindustry transactions tables.

Input	**Output**
(refer to earlier accounting framework for details)	
Cost of domestic materials consumed, whether produced by the primary producing industry or by others as a secondary product, in producer's value	Total value of primary product shipments, in producer's value
	Primary products
Cost of competitive imports consumed, domestic port value	Contract work performed
Secondary products of other industries that are primary to this industry (i.e., secondary products transfers-in, producer's value)	Miscellaneous receipts (Scrap and salable refuse, repair work, research and development, other)
Competitive imports transfers-in, domestic port value	Secondary products of other industries that are primary to this industry (i.e., secondary products transfers-in), in producer's value
	Competitive imports transfers-in, domestic port value
Cost of noncompetitive imports consumed, foreign port value	Shipments of secondary products (i.e., secondary products transfers-out, that is, transferred to industry of primary classification)
Value added	Electric energy sales
Net inventory change, wherever held	Net inventory change, wherever held
Total Gross Input	Total Gross Output

After achieving sectoral material balances in this manner, *technological* coefficients are numerically derived by using two alternative modifications of the *industry* output level. The first, termed *gross domestic output base,* is obtained by simply adding secondary products transfers-in (i.e., secondary products of all other industries that are primary to the consuming industry in question) *to* the total domestic output of the consuming *industry* (i.e., as defined in the earlier sectoral accounts given above). The second, termed *total gross output base,* is obtained by adding to the total domestic output of the consuming *industry* not only secondary products transfers-in but also the competitive imports (domestic port value) competing with the primary products of the industry in question. In both the 1947 and 1958 input-output studies in the United States, *gross domestic output base* was used for each sector in numerically deriving the *technological coefficients* matrices.

The inescapable conclusion to be reached here is that the published *technological* coefficients matrices (i.e., A-matrices) for the U.S. economy for both 1947 and 1958 are subject to serious errors, directly as a result of the modifications described above that are made to both inputs and outputs. Clearly, the purpose of these modifications has been to achieve and demonstrate sectoral material balances; therefore, showing such sectoral balances in supplementary tables should not in general be a matter of any great concern. However, letting the process of achieving sectoral balances interfere with the numerical derivation of *technological* coefficients should be seriously questioned. In effect, observed measurements of reality (i.e., *industry* input functions) are dismissed in favor of a make-believe version of reality (i.e., *made-up* product input functions) that is then represented as being more in tune with theory. Unfortunately, the published multiplier or inverse Leontief matrices for the U.S. economy have reduced utility, since every element in them is potentially made subject to error that is compounded along the way:

To begin with, every element in the Leontief matrix is made subject to error, which means that its determinant, as well as every element in the adjoint of the Leontief matrix are subject to error, and finally, every element in the Leontief inverse matrix is made subject to error, since each element in the Leontief inverse is mathematically obtained by dividing the corresponding element in the adjoint of the Leontief matrix *by* the determinant of the Leontief matrix.

I. SUGGESTED APPROACHES FOR THE NUMERICAL SPECIFICATION OF THE TECHNOLOGICAL COEFFICIENTS MATRICES

After having severely criticized the practices that have been prevalent particularly in the United States with respect to the derivation of the *technological* coefficients matrices, we now set forth here alternative methods that can be used easily and effectively in the future. The central question, as before, is how input-output models should be mathematically and empirically structured under circumstances in which the dominant fact is that the basic information is compiled on an establishment basis.

Temporarily ignoring imports, which will be fully taken into account in Chapter IV in a much more expanded version of this discussion, two alternative approaches seem possible. The first one, the *commodity technology* approach, assumes that the amount of commodity i used in making one unit of *commodity* j is the same in all industries. The second, or the *industry technology* approach, assumes that all commodities, whether primary or secondary, produced in a given industry are made by the same process and, therefore, require the same input structure.[69] Neither of these two approaches is perfect; indeed, each has its own particular set of advantages and disadvantages. The *commodity technology* approach is generally preferable, but one must carefully make this preference conditional on the empirical quality of the *commodity technology* matrix that is generated, by means of a few mathematical manipulations, from basic data on which the *industry technology* matrix itself is based. Ideally, one should be able to abandon either assumption or approach and allow the input coefficients for a particular commodity to differ according to the industry in which it is made. This would lead to a technology matrix with n rows and $n \times n = n^2$ columns. Getting back to the same old problem, such a technology matrix cannot be derived, since the commodity input figures are given for industries (i.e., sets of establishments) and not for their individual products. Until such time as the ideal situation becomes an empirical reality, the two approaches we can now explain should serve as close substitutes.

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[69] The terms *commodity technology* and *industry technology* are borrowed from Richard Stone and his associates. See Cambridge, Department of Applied Economics, *Input-Output Relationships, 1954-1960,* No. 3 in *A Programme for Growth* (London: Chapman and Hall, 1963 and Cambridge, Mass.: The M.I.T. Press, 1963), pp. 11-15.

1. The Derivation of the *Commodity Technology* Matrix

The first problem here, then, is how to generate a *commodity technology* matrix from basic flow data underlying the *industry technology* matrix (i.e., the basic intersectoral transactions matrix which is organized in terms of *product-to-industry* flows). This is obviously the same as the redefinition method of solving the secondary products problem, mentioned earlier.

As a start, let us define the *product-to-industry* flows matrix as $\overline{X} = [\overline{x}_{ij}]$ and the *industry* output levels as a diagonal matrix $\hat{\overline{x}} = [\overline{x}_j]$. Then the *industry technology* matrix can be derived as $\overline{A} = [\overline{x}_{ij}]\ \hat{\overline{x}}^{-1} = [\overline{a}_{ij}]$, where $\overline{a}_{ij}$ represents the amount of commodity i required by *industry* j per unit of its output of primary and secondary products.

Let us further introduce the *make* matrix $\mathring{X} = [\mathring{x}_{kl}]$, where $\mathring{x}_{kl}$ shows the amount (value) of product l made by industry k (i.e., rows indicate industries, while columns indicate products or commodities). In the *make* matrix, the diagonal elements represent the *primary* product output of each industry.

Corresponding to the *make* matrix is the *industry mix* matrix, $U = [u_{kl}]$, where u_{kl} represents the portion of the total amount (value) of product l made by industry k; that is,

$$u_{kl} = \frac{\mathring{x}_{kl}}{\sum\limits_{k=1}^{n} \mathring{x}_{kl}} \qquad k,l = 1,2,\ldots,n, \tag{2.3}$$

where

$$\sum_{k=1}^{n} u_{kl} = 1. \tag{2.4}$$

Using only the *industry technology* matrix and the *make* matrix, and assuming that the input structure of a given product made by an industry is the same as the input structure of the industry which is primarily engaged in the production of that product, we can estimate a *commodity technology* matrix as follows.

The first row in the unknown *product-to-product flows* matrix would be given by

$$
(2.5) \qquad \begin{pmatrix} x_{11} \\ x_{12} \\ \cdot \\ \cdot \\ \cdot \\ x_{1n} \end{pmatrix} = \begin{bmatrix} \mathring{x}_{11} & \mathring{x}_{12} & \cdots & \mathring{x}_{1n} \\ \mathring{x}_{21} & \mathring{x}_{22} & \cdots & \mathring{x}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathring{x}_{n1} & \mathring{x}_{n2} & \cdots & \mathring{x}_{nn} \end{bmatrix} \begin{pmatrix} \bar{a}_{11} \\ \bar{a}_{12} \\ \cdot \\ \cdot \\ \cdot \\ \bar{a}_{1n} \end{pmatrix}
$$

where $x_{11}, x_{12}, \ldots x_{1n}$ in the left hand vector represents the computed or estimated amounts of commodity 1 used in the production of commodity 1, 2, ..., n. Obviously, through expansion, we can compute the new *product-to-product flows* matrix as

$$
(2.6) \qquad \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} \mathring{x}_{11} & \mathring{x}_{12} & \cdots & \mathring{x}_{1n} \\ \mathring{x}_{21} & \mathring{x}_{22} & \cdots & \mathring{x}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathring{x}_{n1} & \mathring{x}_{n2} & \cdots & \mathring{x}_{nn} \end{bmatrix} \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1} \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & \bar{a}_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & \bar{a}_{nn} \end{bmatrix},
$$

or more compactly as

$$
(2.7) \qquad [x_{ji}] = [\mathring{x}_{kl}] \; [\bar{a}_{ji}]
$$

where $[x_{ji}]$ represents the transposed *product-to-product flows* matrix that is estimated indirectly, $[\mathring{x}_{kl}]$ is the *make* matrix, and $[\bar{a}_{ji}]$ is the transposed *industry technology* matrix.

Since, from the *make* matrix, we can easily determine the total domestic output of each product as

$$(2.8) \qquad \mathring{x}_k = \sum_{k=1}^{n} \mathring{x}_{kl}$$

by summing over all industries, we finally arrive at

$$(2.9) \qquad A = [x_{ji}]' \quad \hat{\mathring{x}}^{-1}$$

where A is the derived *commodity technology* matrix, the prime denotes transposition, and where $\hat{\mathring{x}}^{-1}$ is the inverse of the diagonal matrix containing domestic product output levels.

If the method outlined here is used in a straightforward, mechanical way, it may in some cases generate negative input coefficient vectors. When this occurs, it may be necessary to conduct special studies on the individual sectors with negative input coefficient vectors, since negative input coefficients cannot be allowed in the system.

An alternative approach for the derivation of a *commodity technology* matrix has been given by Richard Stone and his associates,[70] in which the *commodity technology* matrix is obtained directly from the *product-to-industry flows* and *make* matrices. In terms of the notations used here, their formulation results in

$$(2.10) \qquad A = [\bar{x}_{ij}] \; [\mathring{x}'_{kl}]^{-1}$$

where A is the derived *commodity technology* matrix, $\bar{x} = [\bar{x}_{ij}]$ is the *product-to-industry flows* matrix, and where $(\mathring{x}')^{-1}$ represents the inverse of the transposed *make* matrix.

2. The Derivation and Use of the *Industry Technology* Matrix

The *industry technology* matrix can be developed and used in input-output analysis as an alternative to the *commodity technology* matrix that can be derived only indirectly as described above. Using the *industry technology* matrix $\bar{A}$, we have

$$(2.11) \qquad X = \bar{A}\,\bar{X} + Y$$

[70] *Ibid.*

where X is the total domestic product output vector, $\bar{X}$ is the total domestic *industry* output vector, Y is the final demand vector, and $\bar{A}$ is as already described.

We know, by using the *industry mix* matrix $U = [u_{kl}]$ that

$$\bar{X} = UX \tag{2.12}$$

which says, in words, that the industry output levels can be found by post-multiplying the *industry mix* matrix U by the vector of total domestic product output levels.

Substituting (2.12) into (2.11), we have

$$X = \bar{A}\,UX + Y \tag{2.13}$$

or

$$(I - \bar{A}U)\;X = Y \tag{2.14}$$

or

$$X = (I - \bar{A}U)^{-1}\,Y. \tag{2.15}$$

One important consequence of this formulation is that the number of commodities need no longer be equal to the number of industries. That is, rectangular *technological* coefficients matrices can be developed and used, so long as the $\bar{A}$ and U matrices are conformable for multiplication.

J. CONCLUDING REMARKS

Perhaps the first, but not necessarily the most significant, conclusion that should be drawn from the preceding discussion in this chapter is that input-output analysis is both conceptually and empirically much more complicated and difficult to perform than is generally appreciated by most economists. It is to be hoped that this chapter has provided an adequately detailed appraisal of some of the most basic measurement problems faced in input-output analysis, so that these *hidden* dimensions of the subject are generally better understood and appreciated.

Secondly, it should be made clear that while theoretically input-output models are capable of fruitful extensions in many directions, experience at least in this country indicates that empirical input-output model construction or analysis has so far been pretty much a victim of circumstances, largely due to nearly insurmountable data problems. Fitting theory to reality has been painful and the results have been far from satisfactory. It is crystal clear that the bag-full of conventions or procedures that have been used in the past to overcome some of the more difficult of these problems have not led to any real solutions. Some of these methods, such as the transfer technique of handling the secondary products problem or the use of the *gross domestic output base* for each sector in the derivation of the *technological* coefficients matrix, etc., have gained legitimacy through repeated use. Fundamentally, these methods amount to no more than elaborately contrived schemes to bring about sectoral balances in the input-output table. Numerical measurements resulting from input-output research, such as the *technological* coefficients matrix, and the inverse of the Leontief matrix, should not have been so deliberately distorted in the process and as a consequence of achieving sectoral balances in the system. Thus, the errors, or rather the obvious misrepresentation of actual economic measurements, inherent in the published U.S. input-output matrices for 1947 and 1958 should not go unquestioned. It seems to this writer that corrective measures should be adopted immediately, particularly before the publication of the results of the 1963 Input-Output Study.

Thirdly, the two alternative methods suggested here, based on *product technology* and *industry technology* assumptions, offer two viable approaches in input-output model construction that should be widely adopted and used. The properties of these two alternative approaches are thoroughly investigated in the series of experiments reported in Chapter IV.

The problem of joint products has not been investigated in detail here and no particular solutions are offered in this thesis. Clearly, this is an area that requires special investigation in the future. A recent contribution made in this area is represented by an unpublished note by Paelinck, who has proposed an accounting scheme where joint products are treated both as an export and as a noncompetitive import.[71]

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[71] Jean Paelinck, "note sur une solution au probleme des produits techniquement joints" (Cambridge, Mass.: Massachusetts Institute of Technology, October 28, 1968, mimeographed, 2 pp.).

In the final analysis, it seems strange that the U.S. Government should sponsor major input-output studies of the U.S. economy without, at the same time, taking actions to make the census of manufactures and other data collection efforts of the government responsive to input-output data requirements. Extending the efforts of the U.S. Bureau of the Census for continual and routine collection of data on intersectoral commodity flows, where the data are organized in *product-to-product flows* terms, would go a long way in alleviating many of the difficulties presently faced in empirical input-output model construction. This seems absolutely necessary, particularly in view of the U.S. Government's present scale of commitment to input-output research, as exemplified by the Interagency Growth Project and the on-going input-output research work located in the United States Office of Business Economics. In addition, there is a considerable volume of input-output related economic research work sponsored by various branches of the U.S. Government at universities, nonprofit research institutions and other research organizations to more than justify the necessary improvements and extensions in the data collection functions of the U.S. Bureau of the Census.

CHAPTER III

PAST EXPERIMENTS ON INPUT-OUTPUT MODELS AND THE SETTING OF THE PROBLEM

A. INTRODUCTION

The previous chapter, which contains a critical study of the empirical structure of input-output models and makes suggestions for improved input-output formulations in the future, provides a necessary background for this chapter, which is intended to serve a three-fold objective: to review the most important findings of past experiments on input-output models, to assess the validity of past experiments by focusing primarily on the empirical structure of the input-output models that they have used, and finally, to describe the set of problems that are investigated in this thesis in the context of past experiments and their findings.

A review of past experiments reveals that, on the whole, they suffer from a number of serious shortcomings. First, it is not meaningful to study the constancy of the *technological* coefficients independently of a given aggregation or *numèraire.* Secondly, the procedure of comparing input-output prediction errors against those obtained by using *naive* methods, a practice all too common to the *predictive* tests, has failed to provide any significant insight into either the temporal constancy of the *technological* coefficients or on the forces that cause changes in them. Thirdly, past experiments generally suffer from a lack of statistical rigor. Regrettable as it may be this is to a certain extent understandable. Usually, previous researchers have had to work with a single input-output model, rather than with a sample of random drawings from a population of such models. Consequently, with rare exceptions, the inferences from the experiments remain highly qualified. Fourthly, and perhaps most importantly, the empirical structure of most of the input-output models used in past experiments is of the *traditional* type, which has been strenuously criticized in the last chapter and is criticized here in more specific terms with respect to particular models used in past experiments.

Despite their shortcomings, past experiments have been useful in a variety of ways. For example, they have underlined the necessity for and have in fact contributed to the development of a variety of methods that can be used to *correct* input-output predictions based on constant *technological* coefficients or to *update* input-output tables to a more recent year in order to prolong their analytical usefulness. Further, special industry studies can and now do play an important role in systematically monitoring and predicting changes in the *technological* coefficients. As a result, the constancy of the *technological* coefficients is no longer as big a handicap or liability as it used to be in input-output analysis.

Rather than conducting one more set of tests on the constancy of the *technological* coefficients, the experiments reported in the next chapter are designed to focus on three problem areas that remain of crucial importance in input-output analysis. The first problem area concerns the sensitivity of input-output predictions to alternative empirical specifications of the *technological* coefficients matrix. These sensitivity experiments, entirely new to the input-output literature, utilize two types of *technological* coefficients matrices for the United States economy for 1958, one based on the measurement of intersectoral flows in *product-to-industry* terms (i.e., *industry technology* matrix) and the other in *product-to-product* terms (i.e., *commodity technology* matrix). These sensitivity experiments, which consist of obtaining predictions to 1961 by using two types of *technological* coefficients matrices and two alternative model formulations, are replicated at six different levels of sectoral aggregation.

The other two problem areas that are studied here relate, respectively, to the structure of input-output prediction errors for different industry groups and to the nature of the *aggregation* content of input-output prediction errors. These experiments differ from earlier attempts in a number of important respects. First, of course, the models used in these experiments contain *technological* coefficients matrices that are both theoretically and empirically superior to matrices of the *traditional* type. Secondly, they are far more comprehensive in scope. Thirdly, they utilize modern statistical methods in drawing inferences, thus rectifying a gap present in most of the past experiments.

The organization of the chapter reflects its three-part objective. The first two sections are devoted to a discussion of the findings in past experiments on the constancy of the *technological* coefficients, the temporal structure of overall input-output prediction errors, the structure of input-output prediction errors for different industry groups, and the effects of aggregation on prediction errors. A criticism of these past experiments, by focusing primarily on the empirical structure of the input-output models on which they are based, is given in the third section. The third section thus serves as a bridge between this chapter and the last. Finally, in the fourth section are described the set of problems to be studied through the experiments in the next chapter, by pointing out their relationships to past experiments, as well as their important differences.

B. TESTS OF CHANGES IN TECHNOLOGICAL COEFFICIENTS

The input-output assumption of constant *technological* coefficients has for a long time been under attack. It is not surprising, therefore, to see that there have been numerous attempts in the past to investigate the validity of this assumption. These studies can be broken into two large groups. The first group includes *direct* tests of changes in *technological* coefficients, either through special industry studies or by comparing *technological* coefficients matrices for different years. The second group comprises of *predictive* tests, in which the general procedure has been to compare conditional point predictions obtained through input-output models with those obtained by using *naive* methods, in order to test for the constancy of *technological* coefficients and, at the same time, to assess the usefulness of input-output models for making predictions.

In general, both the *direct* and the *predictive* tests have been interesting and certainly informative, but their contribution to our understanding of the dynamics of change in *technological* coefficients has been very small. These studies have been most useful in basically two ways. First, the *direct* tests focusing on specific industries have shown that *industry specialists* can contribute quite effectively in large-scale predictive studies in which the input-output framework is used as part of an econometric forecasting model, by pinpointing anticipated technological changes and by specifying how competition between materials is likely to alter the existing input patterns of different industries. Secondly, the *predictive* experiments have underlined the necessity for and have, in fact, led to the development of a variety of methods that can be used either to *correct* input-output predictions based on constant *technological* coefficients or to *update* input-output tables to a more recent year so that their useful life is prolonged. These more recent developments will be mentioned later.

At present, the *constancy* assumption is no longer as big a liability in input-output analysis as it was in the past, since anticipated changes in *technological* coefficients can now be more systematically monitored and entered into the model with the help of *industry specialists.* Still, it is instructive to study these past experiments if for no other reason than to learn from their shortcomings.

1. Direct Tests of Changes in *Technological* Coefficients

Following Hatanaka,[1] the tests that have been conducted by direct observation can be classified into two categories, according to both the methods and the data used in the tests.

In the first category are included a series of studies focusing on time series analysis of changes in the input requirements of specific industries, such as those conducted by Cumberland[2] for a number of critical defense materials (e.g., manganese inputs into steel ingots, tin into tinplate, platinum into electrical equipment, etc.), by Phillips[3] for the anti-friction bearing industry, by Chenery[4] for the natural gas-transmission industry, by Carter (Grosse)[5] for the cotton textile industry, by Ferguson[6] for the air transportation industry, by Smith[7] for the trucking industry, and by Helzner[8] for the steel industry. This group also

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[1] Michio Hatanaka, *The Workability of Input-Output Analysis* (Ludwigshafen am Rhein: Fachverlag fur Wirtschaftstheorie und Ökonometrie, 1960).

[2] J. H. Cumberland, "Examples of Variations in the Behavior of Critical Material Input Coefficients," Interindustry Item No. 17 (Washington, D.C.: U. S. Bureau of Mines, 1952).

[3] Almarin Phillips, "The Variation of Technical Coefficients in the Antifriction Bearing Industry," *Econometrica,* XXIII (October, 1955), 432-441.

[4] Hollis B. Chenery, "Process and Production Function from Engineering Data," in Wassily Leontief, *et al, Studies in the Structure of the American Economy* (New York: Oxford University Press, 1953), pp. 297-325.

[5] Anne P. Carter (Grosse), "The Technological Structure of the Cotton Textile Industry," *ibid.,* pp. 360-420.

[6] Allen R. Ferguson, "Commercial Air Transportation in the United States," *ibid.,* pp. 421-447.

[7] Vernon L. Smith, "Engineering Data and Statistical Techniques in the Analysis of Production and Technological Change: Fuel Requirements in the Trucking Industry," *Econometrica,* XXV (April, 1957), pp. 281-301.

[8] M. L. Helzner, "A Study of Coefficient Variation of Selected Inputs into the Steel Industry," Interindustry Item No. 48 (Washington, D. C.: U. S. Bureau of Mines, 1954).

includes the work of Cameron,[9] who similarly studied the changes in the ratios of real inputs to real outputs in many Australian manufacturing industries. Reviews of these studies, which do not use input-output tables, can be found at varying degrees of detail in Chenery and Clark,[10] Arrow and Hoffenberg,[11] and in Hatanaka.[12]

In the second category are included those experiments in which *technological* coefficients matrices for different years have been compared to find out to what extent the coefficients remain constant over time. This approach has been taken by Leontief,[13] Rasmussen,[14] Sevaldson,[15] the Japanese Government,[16] Iyemoto,[17] and Tilanus.[18]

It is difficult to draw firm conclusions from these two types of *direct* tests, because of a variety of problems associated with them. The conceptual and empirical problems that must be coped with in such tests are of such fundamental nature that they require at least a brief mention.

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[9] Burgess Cameron, "The Production Function in Leontief Models," *The Review of Economic Studies,* XX, 1 (1952), pp. 62-69.

[10] Hollis B. Chenery and Paul G. Clark, *Interindustry Economics* (Fourth Printing; New York: John Wiley and Sons, Inc., February 1965), pp. 162-164.

[11] Kenneth J. Arrow and Marvin Hoffenberg, *A Time Series Analysis of Interindustry Demands* (Amsterdam: North-Holland Publishing Co., 1959), pp.

[12] Hatanaka, *op. cit.*

[13] Wassily Leontief, *et al.,* Studies in the Structure of the American Economy (New York: Oxford University Press, 1953), Chapter II.

[14] P. Norregaard Rasmussen, *Studies in Inter-Sectoral Relations* (Copenhagen: Einer Harcks, 1956), Chapters viii and ix.

[15] Per Sevaldson, "Changes in Input-Output Coefficients," in Tibor Barna (ed.), *Structural Interdependence and Economic Development,* Proceedings of an International Conference on Input-Output Techniques, Geneva, September 1961 (London: MacMillan and Co., Ltd., and New York: St. Martin's Press, 1963) pp. 303-328.

[16] The original report is published in Japanese by the Ministry of International Trade and Industry of Japan. An English summary is given in S. Shishido, "Recent Input-Output Studies in Japan-Particularly in Government Agencies," Stanford Project for Quantitative Research in Economic Development at Stanford University, Memorandum Number C-6 (October, 1957), pp. 10-17.

[17] Hidetaro Iyemoto, "Note on Input-Output Analysis: Differences between the Repercussion Effects in Physical Terms and in Value Terms," *Econometrica,* XXVIII, 3 (July, 1960), 699-700.

[18] C. B. Tilanus, *Input-Output Experiments, The Netherlands, 1948-1961* (Rotterdam: Rotterdam University Press, 1966), pp. 36-51.

a. Tests by Direct Observation: Specific Industry Studies

The study by Cameron of fifty-two Australian manufacturing industries is illustrative of the first category of studies focusing on the direct observation of changes in the input requirements of specific industries. The fifty-two industries studied by Cameron accounted for 6.22 percent of the total Australian work force in the census year 1946-1947.[19] Cameron's conclusions can be divided into two parts, the first pertaining to observed trends in labor inputs and the second concerning the ratios of material inputs to outputs. Both material inputs and outputs were measured in physical terms. Employment, meanwhile, was measured in terms of the number of persons employed, since data on man-hours could not be developed. On material input-coefficients, Cameron concluded that "...they tend on the whole to be approximately constant for short periods of a few years."[20] He found that for half the industries examined, at least the major coefficient remained "...approximately constant for a long period, usually a decade or more."[21] On labor inputs, he observed that "...the level of output and a linear trend factor appears to account for virtually all significant movement in the coefficient in nearly all industries examined."[22]

In studies of this type, it becomes difficult to assess the significance of observed changes in material input coefficients, since a suitable economic model that can be used to test for the significance of the observed changes is, by definition, lacking. In addition, and as Cameron himself points out, such studies usually face two obvious limitations. First, they cannot show whether substitution between factors is possible but can only show whether it has occurred. Similarly, when they show that substitution has taken place between different material inputs, it is difficult to trace it down to relative price changes, quality changes, or simply to technological changes, especially since these are usually intercorrelated. Secondly, such studies are often limited, by necessity, to an analysis of the major material inputs of the industries under study, since data on the less important inputs are often not available.

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[19] Cameron, *op. cit.*, 63.

[20] *Ibid.*, 66.

[21] *Ibid.*

[22] *Ibid.*, 65.

Apart from these limitations, changes over time in the definition and measurement of both inputs and outputs, as well as changes in the extent of statistical coverage, create intractable problems. If both inputs and outputs are measured in value rather than in physical terms, they must be expressed in real terms through price deflation. Developing appropriate price deflators is not an easy task. Further, if both inputs and outputs are measured in physical terms, then they must be defined in great detail, for example at the 7-digit SIC (Standard Industrial Classification) level of disaggregation, since, otherwise, the prospect of aggregation over different units of measurement will have to be faced.

For all these reasons, such studies are relatively limited. If, however, they are conducted within the framework or as a part of a predictive model, so that changes in the material inputs of different industries are studied and predicted by taking into account technological changes, material substitutions effects, etc., then such studies could be extremely useful.

b. Tests by Direct Observation: Comparisons of *Technological* Coefficients Matrices for Different Years

In the second category of studies, focusing on the comparison of *technological* coefficients matrices developed for different years, a series of problems must be faced. First, the input-output tables developed for different years must be made definitionally consistent by adjusting for changes in the SIC content of each sector. Anyone who is familiar with such changes in the United States will readily admit that the results will be far less than perfect. Secondly, the input-output transactions tables must be deflated through the use of reliable price indices.

Neither of these two points appear to have been a source of serious concern to the authors cited earlier. Leontief compared the 14-order United States *technological* coefficient matrices for 1919, 1929, and 1939, while Rasmussen's study focused on an analysis of the Danish matrices for 1947 and 1949. Sevaldson's study compared matrices for 1947 and 1948 for the Norwegian economy. The study by the Japanese government used 36x36 matrices for 1951 and 1954, while Iyemoto compared matrices for 1951 and 1955. Tilanus had at his disposal a continuous series of thirteen 35-order matrices for the Netherlands for the years 1948-1960. There seems to be no clear indication that any of these authors have duly concerned themselves with the first problem. As for the second, only Iyemoto appears to have taken the trouble of deflating his 1955 matrix to the base year 1951, so that both matrices are expressed in the prices of the same year.

The results emerging from these studies are unfortunately not very satisfactory. Iyemoto found that generally the input coefficients remained constant during the period 1951-1955.[23] Leontief prefaced his study with the explicit recognition that actually input coefficients did change during a ten year period.[24] Rasmussen concluded that a number of considerable changes had occurred, but that it did not seem possible to draw any conclusions of a more general nature.[25] Finally, the study conducted by Tilanus, which is by far the most interesting, fails to provide any firm conclusions. In his simple linear regression analysis of each input coefficient as a function of time (with a disturbance term subject to normality conditions, as usual), he finds that the *relative trend* values (defined as the ratio of the regression slope to the intercept, where the intercept is the same as the average input coefficient over the thirteen years) medianwise ranges from 1.5 percent to over 3 percent.[26] Before going any further in his analysis, one may well ask whether the regression slope, which presumably represents the marginal (average) change in a given input coefficient each year, represents any real change or to what extent it veils price changes. Earlier, Tilanus contends that, from a theoretical point of view, "...it is not at all improbable that value coefficients are, on the whole, at least as stable, or even more stable over time than volume coefficients."[27] But, as he points out, whether the value coefficients are actually more stable than the volume coefficients is a matter of price elasticities of demand for inputs by industries. On this question, he provides no empirical evidence in support of his statement. Basically, he shows as evidence his finding that input-output predictions based on value coefficients are, on the whole, better than predictions based on volume coefficients. This, of course, is hardly convincing, as he, himself, recognizes the possibility that "...the additional errors introduced in the prediction by the price indexes applied may have tipped the balance in favor of the value predictions."[28]

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[23] Iyemoto, *op. cit.*, 700.

[24] Leontief, *et al.*, *op. cit.*, p. 27.

[25] Rasmussen, *op. cit.*, p. 130.

[26] Tilanus, *op. cit.*, pp. 44-45.

[27] *Ibid.*, p. 37.

[28] *Ibid.*, pp. 81-82.

2. Predictive Tests of Changes in *Technological* Coefficients and Related Studies

There have appeared in the past a large number of studies that may be conveniently classified as *predictive* experiments. These studies have been conducted for a variety of purposes. While many of them have been interested, for example, in finding out how stable the *technological* coefficients are, others have been merely interested in the predictive performance of input-output models vis-à-vis other, much simpler methods. For all practical purposes, however, these two aims are identical. In fact, many studies focusing primarily on the first objective have also been concerned with the latter, in order to develop certain criteria by which to judge the seriousness of changes in *technological* coefficients. Related to these predictive experiments in substance but different from them in purpose are a second series of studies that have used the input-output system to examine the extent and effects of technological change. A still different and third type of previous research is concerned with the *explanation* of changes in the *technological* coefficients, through econometric analysis. For convenience in discussion, these three types of closely related studies may be somewhat arbitrarily labeled *predictive experiments, studies of technological change,* and *econometric analyses of changes in technological coefficients,* respectively.

Among the *predictive experiments* are included the studies by Leontief,[29] Barnett,[30] Hoffenberg-BLS (U.S. Bureau of Labor Statistics),[31] Selma Arrow,[32] Evans-Hoffenberg-BLS[33]

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[29] Wassily Leontief, *The Structure of the American Economy,* 1919-1939 (2d ed.; New York: Oxford University Press, 1951), pp. 152-159, 216-218.

[30] Harold J. Barnett, "Specific Industry Output Projections," in *Long-Range Economic Projection,* Conference on Research in Income and Wealth, Studies in Income and Wealth, Vol. XVI (Princeton: Princeton University Press, 1954), pp. 191-226. For a revised version of this article by the same author, refer to *Specific Industry Output Projections* (Santa Monica, Calif.: The RAND Corporation, P-208, Revised, 1 September 1951), 43 pp.

[31] No report on this experiment has been published. For a review of the work, refer to Carl F. Christ, "A Review of Input-Output Analysis," in Conference on Research in Income and Wealth, Studies in Income and Wealth, *Input-Output* Analysis: An Appraisal, Vol. XVIII (New York: National Bureau of Economic Research, Inc., 1954), pp. 137-169.

[32] Selma Arrow, *Comparisons of Input-Output and Alternative Projections, 1929-39* (Santa Monica, Calif.: The RAND Corporation, P-239, 14 April 1951), 17 pp.

[33] Since this experiment involved some classified information, the results as well as the details of the method have not been published. For a brief review, refer to Christ, *ibid.,* pp. 165-166.

Hatanaka,[34] and Bailey,[35] as well as those by Clark,[36] Adams and Stewart,[37] Sevaldson,[38] and the Japanese Government,[39] which concentrated on the analysis of the temporal stability of the *technological* coefficients matrices by examining the quality of output predictions only. Ghosh[40] studied both output and interindustry (intermediate) demand predictions, while Matuszewski, Pitts, and Sawyer[41] studied predictions of both intermediate demand levels and competitive import requirements. Also included in this group are the experiments conducted by Rey and Tilanus,[42] and Beerens and Tilanus,[43] who studied the predictive performance of a continuous series of relatively aggregated input-output models available for the Netherlands, by focusing on the quality of intermediate demand predictions.

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[34] Hatanaka, *op. cit.*

[35] William R. Bailey, "An Appraisal of Input-Output Analysis Based on a Documentation of the Interindustry Relations for 1947" (unpublished Ph.D. Thesis, The George Washington University, 1966), pp. 110-151.

[36] H. B. Chenery, P. G. Clark, and V. Cao-Pinna, *The Structure and Growth of the Italian Economy* (Rome: U. S. Mutual Security Agency, 1953), pp. 48-55.

[37] A. A. Adams and I. G. Stewart, "Input-Output Analysis: An Application," *Economic Journal,* LXVI (September, 1956), 442-454.

[38] Per Sevaldson, "Norway," in Tibor Barna (ed.) *The Structural Interdependence of the Economy,* International Seminar on Input-Output Analysis, Varenna, Italy, in the summer of 1954 (New York: John Wiley and Sons, Inc., 1956), pp. 306-307.

[39] Shishido, *op. cit.*

[40] A. Ghosh, *Experiments with Input-Output Models* (Cambridge, England: The University Press, 1964).

[41] T. I. Matuszewski, Paul R. Pitts, and John A. Sawyer, "Alternative Treatments of Imports in Input-Output Models: A Canadian Study," *Journal of the Royal Statistical Society,* Series A, CXXVI (1963), 410-432; and "Inter-Industry Estimates of Canadian Imports, 1949-1958," in W. C. Hood and John A. Sawyer (eds.), *Canadian Political Science Association Conference on Statistics,* 1961, Held at Sir George Williams University, Montreal, Canada (Toronto: University of Toronto Press, 1963), pp. 140-167.

[42] G. Rey and C. B. Tilanus, "Input-Output Forecasts for the Netherlands, 1949-1958," *Econometrica,* XXXI (1963), 454-463.

[43] G. A. C. Beerens and C. B. Tilanus, "Alternative Input-Output Predictions for the Netherlands, 1948-1958," Report 6503 of the Econometric Institute of the Netherlands School of Economics (1965).

These studies, along with some of the others[44] conducted by what might be called the *Dutch* school, have been incorporated into two separate volumes by Theil[45] and Tilanus,[46] published recently.

The second type of experiments, or *studies of technological change* are represented by two separate articles by Carter,[47] who used the 1947 and 1958 input-output tables for the United States in her analyses.

Lastly, the third type of studies are represented by a major attempt by Arrow and Hoffenberg.[48] In their landmark study, they attempted to construct an econometric model aimed at *explaining* the variations of observed industry outputs from those that would have prevailed had the *technological* coefficients remained constant over time. They sought to relate these variations to other economic variables, and, more ambitiously but less successfully, to develop a model that could *explain* changes in interindustry demands to a few major variables.

To summarize, most of the predictive experiments show that input-output predictions are either slightly or in a few cases clearly superior to predictions obtained by using *naive* methods. The superiority of input-output predictions appears to be particularly pronounced

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[44] H. Theil and C. B. Tilanus, "The Demand for Production Factors and the Price Sensitivity of Input-Output Predictions," *International Economic Review,* V (1964), 258-272; C. B. Tilanus and R. Harkema, "Input-Output Predictions of Primary Demand, The Netherlands 1948-1958," Report 6424 of the Econometric Institute of the Netherlands School of Economics (1964); C. B. Tilanus and H. Theil, "The Information Approach to the Evaluation of Input-Output Forecasts," *Econometrica,* XXXII, 4 (October, 1965), 847-862.

[45] Henri Theil, *Applied Economic Forecasting* (Amsterdam: North-Holland Publishing Co. and Chicago: Rand McNally & Co., 1966).

[46] Tilanus, *op. cit.*

[47] Anne P. Carter, "The Economics of Technological Change," *Scientific American,* CCXIV, 4 (April, 1966), 25-31; "Changes in the Structure of the American Economy, 1947 to 1958 and 1962," *The Review of Economics and Statistics,* XLIX, 2 (May, 1967), 209-224.

[48] Arrow and Hoffenberg, *op. cit.*

for the first several prediction years, before the underlying *technological* coefficients matrix becomes relatively *outmoded.* There is sufficient evidence to believe that the *technological* coefficients do not remain constant over time but change, in response to a complex set of interacting forces in the economy, including technological advances, price changes, shifts in the composition of each sector's products, shifts in the sources of supply as between domestic and imported, and a host of other factors.

a. The Rationale of Using *Naive* Predictive Methods

The rationale of the predictive experiments is, in general, to obtain conditional point predictions, using input-output models, and to assess the quality of the predictions by comparing them, *ex post,* with actual observations. Invariably, the quality of input-output predictions are compared with similar results obtained by using *naive* predictive methods, such as the *Final Demand Blowup* and *GNP Blowup* methods and simple regression analysis.

A thumbnail sketch of these alternative *naive* methods can be given as follows. First, the *Final Demand Blowup* procedure is based on the assumption that the ratio of total domestic output (or, alternatively, intermediate demand) to total final demand for each domestically produced homogeneous product remains constant during the prediction period. If we let x^D_{it} be total domestic output of product i and y^D_{it} be total final demand for domestically produced product i at a given base year t, then the extrapolation of the domestic output of sector i τ years ahead is defined as[49]

$$x^D_{i,t+\tau} = \frac{x^D_{it}}{y^D_{it}} \circ y^D_{i,t+\tau}$$

which is simply derived from the proportionality

$$\frac{x^D_{i,t+\tau}}{y^D_{i,t+\tau}} = \frac{x^D_{it}}{y^D_{it}}$$

[49]See Ghosh, *op. cit.,* p. 44, Theil, *op. cit.,* p. 184, and Tilanus, *op. cit.,* p. 61. The formulations given by all three leave out the superscripts "D", which is perhaps a trivial oversight on their part. Since competitive imports can potentially cause confusion on the exact definition of these variables, the superscripts "D" should be employed in order to provide the necessary clarity.

The prediction results would be identical if the basic assumption involved here were stated in slightly different terms; namely, total domestic output of product i will grow at the same rate as total final demand for domestically produced product i. Then, we would have[50]

$$\frac{x^D_{i,t+\tau}}{x^D_{it}} = \frac{y^D_{i,t+\tau}}{y^D_{it}}$$

which would lead to

$$x^D_{i,t+\tau} = \frac{y^D_{i,t+\tau}}{y^D_{it}} \circ x^D_{it} = \frac{x^D_{it}}{y^D_{it}} \circ y^D_{i,t+\tau}.$$

It should be noted that price changes do not effect the extrapolations as long as it is true that there are no divergencies between the price levels of intermediate and of final demand.[51]

Secondly, the *GNP Blowup* method requires no more logical or analytical rigor than is inherent in the *Final Demand Blowup* method: the domestic output of every sector is assumed to grow at the same rate as GNP during the prediction period.

Finally, in the type of regression analysis that is usually employed, the domestic output level of every sector is made a linear function of GNP, using appropriately deflated time series on both variables. Hatanaka, for example, used this approach.[52] Barnett,[53] Selma Arrow,[54] and Ghosh,[55] used *time*, in addition to GNP, as a second independent variable. Taking a somewhat slightly different approach, Ghosh also tried to predict output levels by regression, specifying output as a linear function of final demand.

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[50] Hatanaka, *op. cit.,* p. 169. Hatanaka, too, omits the superscripts "D".

[51] Theil, *loc. cit.*

[52] Hatanaka, *op. cit.,* 205.

[53] Barnett, *op. cit.,* p. 2

[54] Selma Arrow, *op. cit.,* p. 1.

[55] Ghosh, *loc. cit.*

As noted earlier, the predictive experiments, while purporting to test for the temporal stability of *technological* coefficients attempt, at the same time, to assess the usefulness or *workability* of the input-output system for economic forecasting. Thus, usually, the index of agreement between input-output predictions and actual observations is compared with similar indices derived from the application of different methods. In this way, a *criterion of success* is established for the input-output model as a predictive tool, to accept it or to discard it. Not surprisingly, different authors have come to different conclusions on the matter. It is difficult to take these conclusions seriously, since, as Marshall remarked in reviewing Barnett's work, these alternative methods formulated and used were not intended to be legitimate alternatives to the model or procedure being tested, but rather they were on purpose crude and inefficient, almost *reductio ad absurdum* constructions of economic models and forecasting procedures.[56]

One of the few statements in the literature that comes closest to providing a justification for the use of these *naive* methods is given by Tilanus:

> It would be crushing for our input-output prediction model if there were a simple forecasting device requiring less data (and less cost) and yielding better or just as good results. The input-output model should be tested against such a calamity.[57]

Thus, it would seem that, from a statistical viewpoint, the overall (global) prediction error obtained by using any of the *naive* methods provides, at best, a *region of rejection* in testing the null hypothesis of no change in *technological* coefficients. This means that input-output prediction errors equal to or greater than *naive* prediction errors would be relatively improbable of occurrence if the null hypothesis (i.e., no change) were true, but relatively probable given the alternative.

In conclusion, the use of *naive* methods is perhaps acceptable in the context just described, but they certainly provide no help in assessing, *ex post,* significance of the extent of changes in *technological* coefficients. That is, testing the hypothesis of *no change* is instructive, but it is no substitute for determining how serious the changes have been.

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[56] A. W. Marshall, *Comments on H. J. Barnett's "Specific Industry Output Projections"* (Santa Monica, Calif.: The RAND Corporation, p. 243, 1 October 1951).

[57] Tilanus, *op. cit.,* p. 60. Also see Chenery and Clark, *op. cit.,* p. 165.

b. Findings on the Temporal Constancy of the *Technological* Coefficients

Leontief conducted two tests based upon the 1939 input-output tables for the United States, the first involving a 9 x 9 *technological* coefficients matrix and the second a 13 x 13 matrix.[58] Using 1919 and 1929 final demand vectors expressed in 1939 prices, he predicted 1919 and 1929 real output levels, and compared the results with those obtained by using the *Final Demand Blowup* and *GNP Blowup* methods. Using the standard error of prediction[59] as his measure of the overall goodness of fit, he observed that the input-output predictions (backcasts) were substantially better than those obtained by using the *naive* methods. In reviewing Leontief's work, Bodenhorn showed, however, that, according to his calculations based on Leontief's reported results, the standard errors of prediction of the three methods for 1929 were essentially the same.[60]

Barnett,[61] subsequently, predicted 1950 output levels, using a 1939 *technological* coefficients matrix [about 40 x 40 in size][62] and two alternative *full employment* final demand vectors for 1950 projected by Cornfield, Evans, and Hoffenberg.[63] Defining prediction errors in terms of the weighted mean of the absolute values of the errors, he found that input-output predictions for twenty-eight industries[64] rated only second to simple linear regression projections of outputs as a function of GNP and time,[65] while the *Final Demand Blowup* and

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[58] Leontief, *op. cit.* Hatanaka notes that the first test based on the 9x9 matrix was replaced by the second, since another 1939 input-output table became available [see Hatanaka, *op. cit.*, p. 73]. The 13x13 matrix was an aggregated version of the 38x38 matrix for 1939 [see Christ, *op. cit.*, p. 160].

[59] Defined as the square root of the average of the squared differences between predicted and actual total outputs.

[60] G. Diran Bodenhorn, "Review" of Wassily W. Leontief, *The Structure of the American Economy, 1919-1939* (New York: Oxford University Press, Second Edition, 1951), *The American Economic Review,* XLII, 1, (March, 1952), 172-173.

[61] Barnett, *op. cit.*

[62] *Ibid.*, p. 191. Christ notes that Barnett used a 38x38 matrix in his tests [see Christ, *op. cit.*, p. 165].

[63] Jerome Cornfield, W. Duane Evans, and Marvin Hoffenberg, "Full Employment Pattern, 1950," *Monthly Labor Review, LXIV (February-March, 1947), 163-190; 420-432.*

[64] Because of lack of data, Barnett had to delete or aggregate many industries [see Barnett, *op. cit.*, p. 211].

[65] His time series analysis covered the years 1922 to 1941 and 1946.

GNP Blowup projections came third and fourth, respectively. When he compared the unweighted mean of the ratios of prediction errors to the 1939 real outputs, he found that the results were "more equivical."[66] Since the *full employment* final demand projections made for 1950 were not really forecasts but merely two hypothetical estimates – one in which full employment would be attained solely by expanded consumption and another in which full employment would be attained by expanded investment[67] – Barnett's input-output predictions for 1950 contain built-in distortions, and hence, render his comparative results inconclusive.

Next, Selma Arrow, using the same input-output model as did Barnett and actual (observed) final demand vectors, predicted the real outputs for the odd years from 1929 to 1937 inclusive, and compared the results with those attained by applying the *Final Demand Blowup* and *GNP Blowup* methods and with time-series projections of real outputs on GNP and time. Her analysis led to the conclusion that average percentage errors of input-output predictions increase away from the base year (1939), and that they rank only slightly better than the two *blowup* extrapolations. She recognized that the time-series projections were not really projections but simply least square interpolations, since the period of *projections* was part of the period covered in the time series estimation of the regression parameters. Because of this, her time series *projections* should not have been compared with the input-output predictions.

The results were not significantly different in the Hoffenberg-BLS (U.S. Bureau of Labor Statistics) test, which was a replication of Selma Arrow's test, except that time-series projections were omitted. It was concluded that the input-output and *Final Demand Blowup* projections were "approximately equal in quantity," neither being good enough to arouse enthusiasm, whereas the *GNP Blowup* projections were markedly worse.[68]

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[66] Barnett, *op. cit.*, p. 202.

[67] Cornfield, Evans, and Hoffenberg, *op. cit.*, 164-182.

[68] Christ, *op. cit.*, p. 161. As noted earlier, the original work on the Hoffenberg-BLS test has not been published, but was reviewed by Christ. For other reviews, as well as more detail, see Hatanaka, *op. cit.*, p. 78.

Subsequent experiments in the United States, using different versions of the 1947 input-output model, led to somewhat more clearcut results. Among these, the Evans-Hoffenberg-BLS test, which involved the prediction of 1951 real output levels using two different 190-order matrices,[69] has not been made public, since some of the information used in the test was considered classified.[70] A reference is made in Bailey,[71] however, to a test conducted by the BLS in 1952, which could well be the same as the Evans-Hoffenberg-BLS test, using the unaltered 1947 *technological* coefficients. Bailey reports that the BLS compared 1951 input-output predictions for 133 industries with those obtained by using the *Final -Demand Blowup* and *GNP Blowup* methods and found the input-output results clearly superior.[72]

Later, Hatanaka used an aggregated, 64-order matrix in his predictive experiment, focusing on the periods 1937-1940, 1946, and 1948-1950, and concluded that while in the short-range input-output predictions were definitely superior to those obtained by using *naive* methods, he could not say the same for long-range predictions.[73] He noted that this inconclusiveness on long-range predictions was due to "an unsatisfactory testing procedure"[74] forced upon him by the lack of data, concerning time-series projections of output levels on GNP. Nevertheless, his overall conclusion was that "...even in the long-range projection the input-output model maintains an over-all superiority to both the time series projection of output on GNP and the final demand [*Final Demand Blowup*] projection."[75]

In their landmark study, Arrow and Hoffenberg took, as a point of departure, the assumption that the *technological* coefficients are not temporally invariant:

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[69] Apparently, the only difference between these two matrices was that one of them contained some *adjusted* 1951 input coefficients, reflecting changes in the input structure of certain industries [see Christ, *op. cit.,* p. 165].

[70] *Ibid.*

[71] Bailey, *op. cit.,* pp. 117-118.

[72] *Ibid.* Bailey also reports on the analysis of the 1951 predictions obtained by using the 200-order Emergency Model that emerged from the 1947 Input-Output study. The mean weighted prediction error was 13 percent, while the unweighted mean prediction error was 17 percent [*ibid.,* pp. 110-116].

[73] Hatanaka, *op. cit.,* p. 296.

[74] *Ibid.*

[75] *Ibid.*

> ...the assumption of temporally constant input-output coefficients cannot be maintained, since the method based on this assumption has proved inferior in prediction ability to other methods of analysis that are computationally simpler and less revealing structurally.[76]

The scope and content of their study went beyond an investigation of the temporal constancy of the *technological* coefficients, and a complete review of their work will not be attempted here.[77] However, part of their study, in which they had better success, implies still another test of the constancy of the *technological* coefficients, from a somewhat different point of view. In this test, the deviations in the balance equations

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + y_i + u_i,$$

which are the usual balance equations of the open-static Leontief system (apart from random disturbances u_i) computed on the assumption that the *technological* coefficients remained constant at their 1947 value,were fitted to a set of economically important variables.[78] The regressions provided not only tests of the constancy of the *technological* coefficients but also formulas for predicting and correcting the discrepancies in the balance equations resulting

[76] Arrow and Hoffenberg, *op. cit.,* p. 22.

[77] For an adequate review of their work, see Bailey, *op. cit.,* pp. 99-102. At the risk of oversimplification, their work, excepting the constant-coefficients residual analysis discussed above in the text, can be summarized as follows. Their work was focused on the development of two econometric models: the first to predict future numerical values of the *technological* coefficients, and the second to develop a constant-coefficients residual model that could be used to *correct* predictions obtained through straightforward application of the input-output model.

For their variable coefficients model, the authors selected four sectors (lumber and wood products, crude petroleum, petroleum products, and rubber products) which had a small number of major coefficients. They then estimated the structural coefficients of the model by both simultaneous equation and linear programming methods. The simultaneous equations estimates of the structural equations yielded implausible values of the *technological* coefficients. The model was judged to be invalid, as the explanation of changes in the *technological* coefficients by their explanatory variables was not sufficient. The invalidity of their model, however, did not mean that the independent variables chosen were irrelevant to changes in the *technological* coefficients. In fact, they observed, as perhaps a minor consolation, that the residuals in the balance equations derived from the linear programming estimates were smaller on the average than the constant-coefficients residuals. Refer to Arrow and Hoffenberg, *op. cit.,* pp. 127-133.

[78] Such as real defense expenditures on goods and services; wholesale prices, lagged one year; real foreign aid; real per capita disposable income, lagged one year; etc. *[ibid.,* p. 118].

from changes in the *technological* coefficients. Thus, the test for constancy simply involved an analysis of whether or not the constant-coefficients residuals were significantly correlated with the predetermined variables.[79] They found "overwhelming evidence of association between the constant-coefficients residuals and other variables"[80] and, on this basis, concluded that the hypothesis of the constancy of the *technological* coefficients should be rejected.[81] They further concluded, somewhat paradoxically, that while the empirically derived regressions may be very useful for prediction, they "...do not constitute a genuine theoretical explanation."[82]

In experiments conducted in other countries, the results have not been much different on the constancy of the *technological* coefficients matrices: namely, generally speaking, input-output predictions are either only slightly or clearly superior to predictions obtained "...by using *naive* methods. If the *naive* predictions are seriously to construct *rejection regions,*" we would be led to believe in the face of the results that the *technological* coefficient matrices remain relatively constant. It is generally conceded, however, that even when the input-output predictions are superior to *naive* predictions, the *technological* coefficients do not remain constant over time.

Among these experiments, Sevaldson, who tested the 30 x 30 1948 input-output model by backcasting 1947 real output levels, concluded that in comparison with the *naive* methods "the input-output method came out clearly on top, but the lead was not overwhelming."[83] Ghosh, who tested a 47 x 47 1948 model for the United Kingdom by predicting real output levels for the period 1949-1955, concluded that the input-output model does better on the whole than the other, *naive* methods.[84] Likewise, Rey and Tilanus found that the extrapolations based on the *naive* methods are, on the average, inferior to the input-output predictions

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[79] *Ibid.,* p. 117.

[80] *Ibid.,* p. 23.

[81] *Ibid.,* p. 126.

[82] *Ibid.*

[83] Sevaldson, *op. cit.,* p. 307.

[84] Ghosh, *op. cit.,* p. 48. He notes (p. 48) that the input-output model was always more accurate than the *Final-Demand Blowup* method, but less accurate than the regression method only in sectors where final demand comprised over sixty percent of total output, and less accurate than the *GNP Blowup* method only in sectors where final demand comprised over twenty percent of total output.

as long as the latter are not too much *outmoded* compared with the former.[85] The same conclusion is noted in Theil[86] and in Tilanus.[87] Lastly, Matuszewski, Pitts, and Sawyer, who tested the predictive performance of three alternative models each incorporating a different treatment of competitive imports, concluded that although their tests were not conclusive, sufficient evidence seemed to indicate that both import coefficients and *technological* coefficients changed substantially over the period 1949-1956.[88]

C. THE STRUCTURE OF INPUT-OUTPUT PREDICTION ERRORS AND THE AGGREGATION PROBLEM

Apart from testing for the temporal constancy of *technological* coefficients, many of the past experiments have investigated other but closely related problems that bear mention. These problems are best expressed by the following three questions: (1) what is the temporal profile of overall input-output prediction errors (i.e., constant-coefficients residuals), (2) are there any systematic differences in input-output prediction errors for different groups of industries, and (3) what is the effect of sectoral aggregation on input-output predictions? Before presenting the detailed findings of past experiments on these three questions, they can be summarized as follows.

First, input-output prediction errors show an expected temporal profile, generally diminishing toward the base year and increasing away from the base year approximately along a straight line with a positive slope and intercept. The exact shape of such an average error curve, as well as its intercept, are currently clouded by many factors, such as the level of aggregation inherent in the tested model, whether or not the underlying *technological* coefficients matrix is based on *value* or *volume* (i.e., constant dollars) transactions, and a variety of other issues concerning the empirical structure of the *technological* coefficients matrix.

Second, the input-output prediction errors tend to show different characteristics for different groups of industries. For example, industries that sell most of their products to other

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[85] Rey and Tilanus, *op. cit.,* 462.

[86] Theil, *op. cit.,* p. 190.

[87] Tilanus, *op. cit.,* pp. 133-134.

[88] Matuszewski, Pitts, and Sawyer, *op. cit.,* p. 162 ["Interindustry . . ."].

industries seem to come out better in input-output predictions than those that deliver most of their output to final consumption. This matter, however, does not seem to have been studied rigorously. The results just noted are not based on tests of a well structured hypothesis on whether or not there exists a statistically significant correlation between a sector's prediction errors and the portion of its domestic output used by other industries for intermediate consumption.

Third, a few of the past experiments indicate that different aggregations of the same input-output model give different levels of accuracy in prediction. No clear-cut conclusions have been reached, however, on whether or not there exists a statistically significant correlation between average prediction errors and levels of aggregation in input-output models.

1. Findings on the Temporal Structure of Input-Output Prediction Errors.

Among the earlier experiments, Selma Arrow was probably the first to point out that the *technological* coefficients are not constant over time but show a marked time trend, as suggested by the steady convergence of the input-output prediction error indices toward the zero value in the base year (i.e., 1939 in her case).[89] Adams and Stewart, who used a 34 x 34 matrix for 1935 for the United Kingdom to backcast real outputs in 1924, 1930, 1933, and 1934, found that although the percentage discrepancies (errors) for most of the industries fluctuated over the four test years they tended on the whole to diminish toward the base year.[90]

Perhaps the most exhaustive study of the temporal structure of input-output prediction errors has been conducted by what we have earlier called the *Dutch* school. Rey and Tilanus used a series of ten input-output tables for the period 1948-1958, and Tilanus used a series of thirteen tables for the period 1948-1960, focusing, among other things, on the prediction of the interindustry deliveries of twenty-seven sectors, by using input-output tables of each

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[89] Arrow, *op. cit.,* p. 8.

[90] Adams and Stewart, *op. cit.,* 450.

preceding year and observed final demands for each prediction year. They found that for all industries combined, the root-mean squared (RMS) prediction error was almost eight percent if the underlying input-output table was that of the preceding year.[91] When the time difference τ was longer, the RMS prediction error increased about proportionately to the square root of τ.[92] Theil, writing largely on the results obtained by Rey and Tilanus, remarked that, in this latter case, the RMS prediction error increased *less than* proportionately to the time difference.[93] The same conclusions are noted in Tilanus.[94]

2. Findings on the Structure of Input-Output Prediction Errors for Different Industry Groups

Many of the earlier experiments did not contain a systematic examination of the structure of input-output prediction errors by different industry groups. Given the fact that input-output predictions are conditional point predictions in which the final demand vector is exogenously specified, one would expect intuitively such predictions to be much better for industries which deliver a large part of their output to the final demand sectors, since the model would obviously have very little left to *explain* in respect to these particular sectors. Conversely, again on intuitive grounds, it could be argued that one can judge the *efficiency* of the input-output model as a predictor by examining the seriousness of the prediction errors for those sectors that deliver most of their output to other industries for intermediate consumption, since in these cases, the model would have *a lot* to explain. Thus, it is both interesting and important in input-output analysis to see if there is a statistically significant correlation between input-output prediction errors by individual industry sectors and the extent to which industries depend on other industries for their customers.

Ghosh, in his experiments, dealt with the question just raised and, contrary to what one would have intuitively expected, found that in some industries with a high final demand-to-total output ratio, the prediction errors were large. In explaining the reason for this, he surmised that the particular industries involved were relatively small and that, therefore,

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[91] Rey and Tilanus, *op. cit.*, 462.

[92] *Ibid.*

[93] Theil, *loc. cit.*

[94] Tilanus, *loc. cit.*

they were more sensitive to changes in other industries.[95] His study, however, was not so structured as to test a well defined hypothesis in this respect. Hence, his observations should be interpreted with due care.

Ghosh's observation was confirmed by Rey and Tilanus who investigated the same question and found that the predictions for those industries whose interindustry deliveries cover large sums were relatively better than for other industries.[96] In addition, Theil noted that the six largest sectors, which jointly accounted for more than fifty percent of total production for intermediate demand, were characterized by logarithmic prediction errors that were on the average much smaller than those of the other sectors.[97]

In conclusion, there appears to be sufficient evidence to believe, contrary to what one would perhaps intuitively expect, that input-output predictions are better for those sectors whose interindustry deliveries comprise a higher portion of their total output than for the others. The reason for this would be that the industries that have strong forward linkages (i.e., they supply many other industries with their raw materials or intermediate products) are perturbed or affected to a smaller extent by changes in the *technological* coefficients matrix than those industries which have relatively weaker forward linkages. Although the subject suggests itself for more rigorous statistical testing, one should keep in mind the fact that the tests would be rather limited in what they would reveal, since the chances would be relatively high for the results to be biased at the very outset by the type of sectoral classification or aggregation system inherent in the *technological* coefficients matrix used in the tests.

3. Findings on the Effects of Aggregation on Prediction Errors

What is meant by the "effects of aggregation on prediction errors" has been well posed by Theil in the form of a question: "if total output predictions of certain industries are derived by means of an aggregated input-output table, what is the nature of the forecasting error that is to be ascribed to the aggregation?"[98] There now exists a substantial body of literature

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[95] Ghosh, *loc. cit.*

[96] Rey and Tilanus, *loc. cit.*

[97] Theil, *loc. cit.*

[98] H. Theil, "Linear Aggregation in Input-Output Analysis," *Econometrica,* XXV, 1 (January, 1957), 111-122.

dealing with the theoretical and practical problems concerning aggregation in input-output models.[99] Aggregation criteria have been developed to minimize the loss of detail and to prevent the distortion of results that stem from dissimilarity among the individual micro-components aggregated into more broadly defined groups or sectors.

In an early test of the effect of aggregation on prediction errors, Balderston and Whitin found that the 1939 input-output model of the United States, when solved at different levels of aggregation, produced different solutions compared at the same level of aggregation.[100] In another experiment, conducted by the Japanese Government, a comparison was made of real output predictions to 1954 from an 182-order and an aggregated 31-order model constructed for 1951, and it was found that the prediction errors for groups of sectors were almost the same. On this basis, it was concluded that "...errors due to aggregation in most cases cancel each other, and that for input-output analysis the number of sectors need not always be large unless demand for specific commodities must be examined in detail."[101]

Two subsequent experiments seem to contradict the results attained in the Japanese study. First, Bailey reports on a comparison made by the BLS (U.S. Bureau of Labor Statistics) of the predictive performance of a 190-order and an aggregated 50-order model derived from the 1947 Input-Output Study.[102] When prediction errors for 1951 were compared, it was found that the smaller model had a smaller average prediction error, both on a weighted and an unweighted basis. This result prompted Bailey to remark that "the process of consolidation involves a certain averaging of diverse errors which makes it almost inevitable that the smaller classification system should result in a smaller error."[103]

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[99] Refer to the references cited in Chapter I, in the section entitled "The Aggregation Problem in Input-Output Analysis."

[100] J. B. Balderston and T. M. Whitin, "Aggregation in the Input-Output Model," in Oskar Morgenstern (ed.), *Economic Activity Analysis* (New York: John Wiley and Sons, Inc., 1954), p. 104.

[101] Shishido, *op. cit.,* p. 9. Also see Hatanaka, *op. cit.,* pp. 81-82.

[102] Bailey, *op. cit.,* p. 135.

[103] *Ibid.*

Secondly, Ghosh studied the predictive performance of input-output models of the United Kingdom constructed for 1948 at four different levels of aggregation (i.e., 47 x 47, 27 x 27, 24 x 24 and 10 x 10) and observed that "...different aggregations give different levels of accuracy in prediction, though in the case of a number of industries the divergences are small."[104] His predictions were obtained for the period 1949-1955. For 1949, the four different aggregations did not display any significant deviations in average percentage error. For the period 1950-1955, however, the 10 x 10 model gave the smallest, while the 47 x 47 model gave the largest average percentage error (except for 1954). On the other hand, the 27 x 27 model gave uniformly better results (except for 1952) than the 24 x 24 model. When the comparisons were made after consolidating the 47 x 47 model results into the other levels of aggregation, the observed pattern was substantially the same.[105]

It thus appears, in general, that while different aggregations give different levels of accuracy in prediction, no clear-cut conclusion can be reached on the more important question of whether or not there exists a statistically significant correlation between average prediction errors and levels of aggregation in input-output models. To put it somewhat differently, the existence and significance of the *aggregation bias* in input-output analysis do not yet seem to have been given a rigorous statistical test.

D. A GENERAL CRITICISM OF PAST EXPERIMENTS

Perhaps the most important question that must be raised in assessing the validity of the results attained in past experiments concerns the empirical structure of the input-output models used in these experiments. This clearly appears to be an Achilles' heel in nearly every single experiment that has been mentioned or discussed earlier, including *direct* tests of the constancy of *technological* coefficients using input-output matrices constructed for different years.

The empirical structure of the models used in past experiments are criticized by focusing on four areas: (1) measurement of intersectoral relationships, (2) errors of an obvious nature, (3) special measurement problems associated with individual models, and finally (4) problems in developing *goodness of fit* measures that stem directly from the empirical structure of the model used in the experiments.

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[104] Ghosh, *op. cit.*, pp. 59-60.

[105] *Ibid.* Refer particularly to Table 6.5.

1. Measurement of Intersectoral Relationships

To begin with, an examination of the empirical structure of the models used in these experiments reveals a few important facts that immediately cast a shadow on the validity of the results achieved in a majority of the cases. In all experiments using the results of input-output studies in the United States (this includes the studies by Leontief, Barnett, Selma Arrow, Hoffenberg-BLS, Evans-Hoffenberg-BLS, Arrow and Hoffenberg, Hatanaka, and Carter), the underlying model structure is of the *traditional* type that has been so strenuously challenged in the last chapter.

It will be recalled that, among other things, the *traditional* empirical model structure can be identified as one in which the producing sectors are defined in terms of homogeneous product groups and, generally speaking, the consuming sectors are defined as industries (i.e., as mutually exclusive sets of establishments), *and* in which the following generally hold true:

(a) the secondary products of a producing sector are transferred out and shown as an artificial or *fictitious* transaction to the industry where such secondary products would be primary, and then they are distributed to all consuming industries as part of the primary products of that industry;

(b) the by-products and sometimes the scrap products of the producing sectors are assumed to be completely consumed internally just prior to the inversion of the Leontief matrix (i.e., they are included in the intra-industry cells, which are the same as the diagonal elements in the matrix);

(c) the output of each (consuming) industry that is used to compute the *technological* coefficients is defined as the *gross domestic output base,* which consists of the value of primary and secondary products and miscellaneous receipts of that industry *plus* the secondary products of *other* industries that are principally produced by that industry, adjusted for inventory changes.[106]

(d) the competitive imports transfers row, which contains *fictitious* sales from the *competitive imports sector* to each domestic industry, and the noncompetitive imports row

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[106] Alternatively, to the ***gross domestic output base*** for a given industry is added the value of total competitive imports transfers-in (i.e., the value of competitive imports substitutable for the primary products of the industry in question), in order to arrive at the ***total output base*** for that industry.

are retained as two separate rows in both the Leontief matrix and the inverse matrix (when the *total output base* is used in computing these matrices), which leads to redundant entries in the competitive imports transfers row and to errors in all coefficients through the errors thus introduced into the denominators.

As a consequence of the *fictitious* transactions that are created purely for accounting purposes *and* as a result of specifying the output level of each (consuming) industry in a manner that does not in the least represent its actual output level, the *technological* coefficients matrices used in these experiments are made subject to what one is unfortunately forced to call *deliberate* measurement errors. As indicated and documented in detail in the last chapter, the objective of achieving sectoral commodity balances in input-output flow tables is put ahead of the task of obtaining an accurate measurement of the structural parameters (i.e., the *technological* coefficients). This is indeed regrettable, since the input-output model empirically put together in this way is seriously deficient for use in the structural analysis of economic systems. It is, of course, quite baffling that the previous researchers have not addressed themselves to the scientific validity and wisdom of the procedures described here that lie at the root of these *deliberate* measurement errors. In the literature, the measurement problems raised here are either never mentioned at all or are dismissed out of hand after making a few vague references, for example, to the secondary products problem.

2. Errors of an Obvious Nature

Further, the models used in some of the earlier predictive experiments contain a large number of errors (apart from the measurement problems just raised) that should be mentioned. The following quotation from Leontief, in this context, is quite revealing:

> The empirical evidence available at the present time consists of the tables of interindustrial relationships for the years 1919, 1929, and 1939. The first two, results of singlehanded purely exploratory efforts, are very rough indeed, and even the last, although much more comprehensive in its scope, can hardly be considered as representing more than a first approximation to a thorough statistical job that could be done under present day conditions. The large size of the *undistributed* outputs (approximately 25-30 percent of the totals in 1939) gives a fair measure of the deficiencies of the empirical data now available for testing purposes.[107]

[107] Leontief, *op. cit.*, p.216 [*The Structure....*].

As noted earlier, Leontief himself used the 1919, 1929, and 1939 tables aggregated to 13 sectors. The tests conducted by Barnett, Selma Arrow and Hoffenberg-BLS were based on a 38 x 38 version of the 1939 *technological* coefficients matrix.[108]

3. Special Measurement Problems

Thirdly, the models used in the experiments contain certain individual differences as to their empirical structure that need to be discussed in assessing the validity of the experiments themselves. For example, in the *Dutch* experiments (i.e., those using the input-output models constructed for the Netherlands), the *technological* coefficients matrix is defined as the domestic matrix only; in these models, imports are directly allocated to their ultimate destination (i.e., to the buying domestic industries or sectors).[109] In the predictive experiments, therefore, competitive imports are assumed not to be substitutable for domestically produced products and are predicted exogenously.

Another example is provided by the 42-sector 1949 Canadian model used by Matuszewski, Pitts, and Sawyer, in which the diagonal elements of the *technological* coefficients matrix was assumed to be zero.[110] Thus, the output figures derived from the analysis excluded intra-industry consumption and had to be raised to include this production by applying *multipliers* that were apparently developed with this specific purpose in mind.[111] This practice seems reasonable except that in all likelihood these *multipliers* or *adjustment coefficients* were computed for 1949 and were then assumed to remain constant between 1949 and

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[108] *Ibid.*, 216-218. Complete specifications of the model are not given. In an earlier test, Leontief predicted the 1929 real output levels, using a 9x9 1939 *technology* matrix and 1929 final demand vector. Hatanaka indicates [see, Hatanaka, *op. cit.*, p. 73] that in this particular input-output table, inputs for current production were not separated from the new investment inputs. Apparently, this first test was replaced by a second test (indicated above), after another 1939 input-output table became available in which the inputs for current production were distinguished from the new investment inputs. Also see Leontief, *op. cit.*, p. 154.

[109] The Netherlands Central Bureau of Statistics, *Input-Output Tables for the Netherlands*, Statistical Studies, No. 16 (Zeist, Netherlands: Uitgeversmaatschappij W. de Haan N. V., July, 1963), p. 6.

[110] Matuszewski, Pitts, and Sawyer, *op. cit.*, 144, footnote 5 ["Inter-Industry . . ."].

[111] *Ibid.*

1956, the terminal year in the experiments. A further point that should be raised is that unallocated inputs and outputs that were present in the 1949 table were left out before inverting the Leontief matrix. The authors were then left in the uncomfortable position of assuming that unallocated outputs in all subsequent years remained constant at the 1949 level.[112] Moreover, the authors met with a series of statistical difficulties that could not be helped. The 1949 interindustry transactions table was subject to error, particularly because of the lack of information on the components of the distributive margins between producer's and purchaser's values.[113] Originally, the model was constructed in terms of purchaser's prices.[114] Also, there was a problem in estimating the goods and non-factor services, other than materials, fuel, and electricity, used by manufacturing industries.[115]

The direct estimates of the *actual* 1956 outputs and of competitive imports, which were used in judging the quality of the predictions, were "probably" subject to error.[116] No independent estimates of the output levels of the four service industries could be made for 1956 and they were thus excluded from the analysis of prediction errors.[117] Finally, a still different type of factor affecting the interpretation of the results was connected with the choice of the years 1949 and 1956 as the years of comparison. While 1949 was characterized by the presence of some import restrictions and by the fact that a ten percent devaluation of the foreign exchange rate took place in the same year, 1956 was characterized as a year in which there were no restrictions on foreign exchange and in which there was also a freely fluctuating exchange rate in place of a fixed rate.[118] Also, while 1949 was a year of moderate, if not slow, economic growth, 1956 was the peak year "...of a boom in Canadian resource industries which strained the productive capacity of many domestic industries, requiring domestic output to be supplemented by imports in many cases."[119]

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[112] *Ibid.*

[113] *Ibid*, 164.

[114] John A. Sawyer, "The Measurement of Inter-Industry Relationships in Canada," *The Canadian Journal of Economics and Political Science*, XXI, 4 (November, 1955), 485.

[115] Matuszewski, Pitts, and Sawyer, *loc. cit.* ["Inter-Industry . . ."].

[116] Matuszewski, Pitts, and Sawyer, *op. cit.*, 422 ["Treatments of Imports . . ."].

[117] Matuszewski, Pitts, and Sawyer, *op. cit.*, 150, footnote 6 ["Inter-Industry . . ."].

[118] Matuszewski, Pitts, and Sawyer, *op. cit.*, 425, ["Treatments of Imports . . ."].

[119] *Ibid.*

A further example is provided by the 47-sector input-output model of the United Kingdom for 1948 that was originally constructed by Stewart[120] and later used by Ghosh[121] in his experiments. In this model, as in the Canadian example, current transactions between establishments classified in the same industry have been *netted* to zero (i.e., the diagonal entries in the *technological* coefficients matrix are zero).[122] In addition, intersectoral transactions are recorded not in producer's values, which is the standard method, but in purchaser's values.[123] Each entry in the table, therefore, includes not only the *real* transaction between any two pairs of sectors but also the distributive margins associated with that transaction. Since the table is, moreover, constructed along *traditional* lines as indicated earlier, the *technological* coefficients matrix unfortunately contains serious measurement errors. To make matters worse, the *technological* coefficients matrix is not symmetrical with respect to the *margin* industries. For example, Row Sector 47 is *Services,* while Column Sector 47 is *Rail Transport.* Column Sectors 48 (*Road Transport*) 49 (*Other Transport and Distribution*), and 50 (*Other Services*) are completely left out from the 47 x 47 structural matrices.[124]

4. Problems Concerning *Goodness of Fit* Measures

In all past experiments using input-output models in which the underlying empirical model structure is of the *traditional* type (this includes all experiments using U.S. input-output models), the *goodness of fit* measures must be scrutinized with special care. Essentially, the problem consists of developing accurate estimates of the *actual* sectoral output levels for the prediction years, in order to derive *goodness of fit* measures, *ex post,* for the predictions.

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[120] I. G. Stewart, "Input-Output Table for the United Kingdom," *The Times Review of Industry, London and Cambridge Economic Bulletin,* New Series, No. 28 (1958).

[121] Ghosh, *op. cit.,* p. 148.

[122] *Ibid.,* Supplementary Tables B.1 and B.2.

[123] *Ibid.,* Supplementary Table B.1., General Notes, No. 2.

[124] *Ibid.,* Supplementary Tables B.1 and B.2.

In such experiments, input-output predictions of output levels definitionally include, for each producing sector, the value of a given domestically produced commodity (commodity group), *wherever made* (i.e., produced by the primary producing *industry* as its primary product, and, by other industries as a secondary product) plus the secondary products of the primary producing *industry* and its miscellaneous receipts, adjusted for inventory changes. Observed or actual output levels, on the other hand, are measured for inter-census years in the United States in two ways: (a) the value of a commodity (commodity group) *wherever made,* which is called *primary product output,* and (b) the total value of shipments of a given *industry* (i.e., the value of its primary and secondary products), which is called *industry* output. Since the *primary* and *secondary* products of a given *industry* are separately available only for the census years (e.g., 1954, 1958, 1963), it is not possible to obtain direct estimates of sectoral output levels for inter-census years that are consistent with *traditional* input-output definition of sectoral output as just noted. Since the *predicted* and *observed* output levels are thus not consistent definitionally for the inter-census years, special estimates are developed of *actual* output levels that are consistent with input-output definitions. This is accomplished by *updating* base year output levels defined in input-output terms to the respective prediction years by using production indices,[125] particularly the FRB (Federal Reserve Board) Index of Industrial Production.

The Federal Reserve Board of Governors publishes its index of industrial production in two groupings. The first is a grouping of production into durable and nondurable manufactures, each broken down into broad industrial categories (e.g., primary and fabricated metals, machinery, etc.). The second is a grouping of production by major markets (i.e., consumer goods, business and defense equipment, materials and intermediate products), which in turn are further broken into more refined categories (e.g., automotive products, home goods,

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[125] For example, the Fabricant Output Indices and Census (U.S. Bureau of the Census) Production indices.

Fabricant published his adjusted and unadjusted indices of physical output for census years in the period 1899-1937. Later on, he computed, but did not publish, detailed indices of 1939 output relative to 1937. His adjusted output indices including those for 1939, have been published, without the details, in Solomon Fabricant, *Employment in Manufacturing 1899-1939* (New York: National Bureau of Economic Research 1942), pp. 264-331.

For further detail on the Fabricant and other indices, refer to Arrow and Hoffenberg, *op. cit.,* pp. 187-192.

consumer staples, etc.).[126] The FRB indices are based primarily on physical output data for fairly homogeneous products and are weighted by value added data for the base period to obtain measures for each product category. Even if one ignores the deficiencies of the FRB indices, it is clear that the practice of using them to *update* base year input-output-defined sectoral output levels is questionable, given the fact that, generally speaking, FRB indices and input-output-defined sectoral output levels are, on the whole, definitionally mismatched. Further, in such an adjustment process, the implicit assumption of a constant ratio between secondary products transfers-out *and* total primary product output in each sector cannot be readily accepted without convincing empirical evidence.

Finally, the base year input-output-defined sectoral output levels, after having been *updated* to the respective prediction years, must be deflated back to base year prices. This would be necessary if the predictions are expressed in base year prices, as they are theoretically thought to be. Even if fairly good price indices are available, the deflated results would still be far from perfect, since, again, the product composition of the respective indices would not be well matched against the product composition of input-output-defined sectoral output levels.[127]

E. DEFINITION OF THE PROBLEM AREA TO BE STUDIED IN LIGHT OF PAST EXPERIMENTS

Perhaps the most important lesson to be gained from the critical review of past experiments given above is that the results can be no better than the empirical quality of the model used in the experiments. More to the point, the *traditional* empirical model structure must be rejected in all future experiments of the type discussed above, if the results are to meet minimum scientific standards.

As pointed out in the last chapter, the *technological* coefficients matrix should ideally reflect the input structure of each product (homogeneous product group). Because of the

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[126] For a complete discussion see Board of Governors of the Federal Reserve System, *Industrial Production, 1959 Revision* (Washington, D. C.: 1960); and *Industrial Production, 1957-1959 Base* (Washington, D. C.: 1962).

[127] Note that each input-output-defined sectoral output level is inclusive of the *secondary* products of the primary producing *industry;* and this is the source of the problem raised here.

establishment unit of industrial classification used in many countries, this ideal can only be approximated in most cases by a *technological* coefficients matrix in which each vector represents the input structure of an *industry* (i.e., a set of establishments).[128] In both cases, the producing sectors (rows) are defined in product terms.

To summarize, two models are formulated in the next chapter. The first (Model 1) is based on the *commodity technology* assumption (i.e., a given product produced in different industries has the same production function), while the second (Model II) is based on the *industry technology* assumption (i.e., different products produced by a given industry have the same production function). These two models are formulated first with competitive imports treated endogenously and secondly with competitive imports treated exogenously.

The exogenous versions of the two alternative models are then used in the experiments. The models are empirically formulated, with 1958 as the base year, at four different levels of sectoral aggregation (i.e., 79 x 79, 60 x 60, 45 x 45, and 17 x 17), and conditional point predictions of intermediate demand and domestic product output levels are obtained for 1961. These results are used in studying the comparative predictive performance of the two alternative models, the effects of aggregation on aggregate model prediction errors, and the structure of detailed (sectoral) prediction errors.

The three types of experiments just noted may be conveniently called the *sensitivity experiments, experiments on the aggregation problem,* and *experiments on the structure of prediction errors.* The sensitivity experiments are entirely new to the input-output literature, and the results of these experiments are likely to have significant impact on the empirical construction of input-output models in this country and abroad. These results should also contribute to a better understanding of the structure of input-output models and their application in economic analysis and forecasting and in programming for economic development.

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[128] As a slight qualification, it must be mentioned that some non-manufacturing sectors (as consumers) are defined in *activity* terms, which approximates a pure product definition.

The experiments on the structure of prediction errors and the experiments on the aggregation problem differ from earlier efforts in at least two fundamental ways. First, the *technological* coefficients matrices used in these experiments are defined both in *product-to-industry* and in *product-to-product* terms, which represents an important departure from past experiments using traditionally defined *technological* coefficients matrices that have been criticized heavily in this chapter and the last. Secondly, the present experiments are far more comprehensive, in that many more issues are investigated under alternative sets of conditions. In conclusion, the experiments reported in the next chapter should, taken together, contribute substantially to a better understanding of the structure and application of input–output models.

F. CONCLUDING REMARKS

The subject of this chapter has been, essentially, to place the set of problems studied in this dissertation within the historical and scientific context of past experiments on the structure of input-output models. The basic conclusion of this chapter is that past experiments have groped with a few rather important issues in input-output analysis, such as the constancy of the *technological* coefficients, the structure of prediction errors, and the effects of aggregation on prediction errors, but have failed to reach conclusive and operationally useful results on any of these major issues. The reasons for this have not been altogether unexpected.

First, the question of the constancy of the *technological* coefficients over time cannot be studied independently of the level of aggregation inherent in the model. Therefore, inferences made on the constancy of the coefficients at different levels of aggregation in different studies must be taken with a great deal of caution, even if these studies are fairly well designed in every other respect.

Secondly, one finds that the experimental design of past studies leave a great deal to be desired. The procedure generally followed in the *predictive* tests, which make up the bulk of past experiments, has been to compare conditional point predictions obtained from input-output models with those generated by using *naive* methods, with the latter serving as the *rejection region.* This is most unsatisfactory, since it would be difficult to accept an hypothesis of *no change* in the *technological* coefficients if input-output predictions did better and still more difficult to discard the input-output framework for making predictions if the *naive* predictions did better. There are now ways of *updating* past input-output tables to a more recent year so as to prolong their analytical utility and there exist a few methods for correcting input-output prediction errors. Especially with the increasing involvement of *industry specialists* in predicting changes in the *technological* coefficients, within the framework of large-scale econometric studies designed for short and long term multi-sectoral economic forecasting, one cannot take too seriously some of the findings of these past experiments that show that *naive* methods do better as predictors than the input-output model. In fact, one can say that these recent developments have stemmed from a recognition of the shortcomings that have been the subject of these past experiments focusing on the constancy of the *technological* coefficients.

Thirdly, and in a somewhat broader sense than just indicated, most of the past experiments suffer from a lack of statistical rigor, for example in testing for systematic variations in input-output prediction errors for different industry groups or in testing for the effects of aggregation on prediction errors. With only one or two available models to work with, there are certainly limitations on the extent to which the researcher can utilize modern statistical methods in his analysis. Nevertheless, a general lack of statistical orientation is quite evident in most of the past experiments.

Fourthly, and perhaps the most serious of all, past researchers have not paid sufficient care to the empirical structure of the input-output models they have used in their experiments, which, by itself, raises serious questions on the validity of their results in most instances. One is struck in these experiments not necessarily by a display of naivetè in the unhesitating use of input-output models, with no searching questions asked on what they actually represented empirically and theoretically, but rather by an acquiescent attitude which cannot be condoned.

The input-output model now plays a far more significant role in economic analysis, multi-sectoral economic forecasting, and in programming for economic development than ever before. Similarly, the results obtained from studies in which the input-output system plays a central part are being increasingly used in shaping important public policy decisions, as well as in planning for corporate development.

Given these trends, and in the light of the four major conclusions noted above, there is currently an obvious need for new experiments that are designed to focus on certain key problem areas in input-output analysis that still remain by-and-large unresolved. Three problems are studied in the next chapter. The first is concerned with the sensitivity of input-output predictions to alternative empirical model specifications, constructed on the basis of *commodity technology* and *industry technology* assumptions. The results of these sensitivity experiments, which will be completely new to the literature, should be of considerable significance in the construction and application of input-output models. The second and third problem areas, respectively, involve the structure of input-output prediction errors for different industry groups and the effects of aggregation on prediction errors. Further results in these two areas should, likewise, contribute to a better understanding of the structure of input-output models and should enhance their usefulness in practical applications.

CHAPTER IV
DESIGN AND ANALYSIS OF EXPERIMENTS

A. INTRODUCTION

The purpose of this chapter is to report on the design and analysis of the experiments that have been conducted to investigate a number of questions. These questions can be listed quite broadly, as follows:

1. How do the *commodity technology* and *industry technology* models compare in terms of their predictive performance at each level of sectoral aggregation, as well as at all levels of aggregation taken together?

2. What is the empirical nature of the relationship between the level of sectoral aggregation inherent in a model and its predictive performance? How do the two models compare in this respect?

3. Do input-output prediction errors show systematic variations, and if so, can meaningful inferences be drawn about such variations?

In the first part of the chapter, the *commodity technology* and *industry technology* models are mathematically derived. This is followed by a theoretical discussion of the structure of input-output prediction errors. Next, a detailed description of the experiments is given, followed by ananalysis of the results. The detailed conclusions reported earlier are summarized in the last part of the chapter and suggestions are made for further research.

B. THE MATHEMATICAL STRUCTURE OF THE MODELS

The mathematical structure of the models developed here is to a certain extent conditioned by the empirical content of the 1958 Input-Output Study of the United States. Some empirical properties of the 1958 Study have already been thoroughly examined and severely criticized in Chapter II.

For the present purposes, the following features of the 1958 Study, which may or may not have been noted earlier, should be singled out not for further criticism but simply for emphasis:

(1) The 1958 *technological* coefficients matrices for the United States as constructed in the 1958 Study and as subsequently published are based on *traditional* methods of measurement, with the consequence that the empirical structure of these matrices is seriously deficient. This major point has already been so extensively and strongly made both in Chapters II and in Chapter III that nothing more remains to be said here. Completely new *technology* matrices must be *derived* from the basic information on which the 1958 Study is based. The *commodity technology* and *industry technology* matrices developed and used in the experiments here are new and differ both theoretically and empirically from the published 1958 *technological* coefficients matrices.

(2) The *technological* coefficients matrix (i.e., representing either *commodity technology* or *industry technology*) that can be *derived* from the 1958 Study represents a *combined* matrix. That is, the two matrices A^D and M, which respectively denote the *domestic technology* matrix and the *direct competitive imports requirements* matrix, are not observed separately but only as a *single* matrix (i.e., $(A^D + M)$, with M *embedded* in the observed *technology* matrix).

(3) The final demand sectors in the 1958 Study consist only of final demand for domestically produced goods and services. That is, the final demand sectors are shown as purchasing all competitive imports from a single source (i.e., Row 80A in the published tables). This procedure has the undesirable consequence that the product composition of competitive imports delivered to the final demand sectors is not known and cannot be determined from the information given in the Study itself.

(4) Noncompetitive imports are shown as being purchased from one source (i.e., Row 80B in the published tables). This procedure, again, has the undesirable consequence that the product composition of the noncompetitive imports is not known and cannot be determined from the information given in the 1958 Study itself.

In view of these four points, two alternative models are developed that make maximum use of the information available from the 1958 Study. The *first* is based on the hypothesis of *commodity technology* (Model I) and the *second* on the hypothesis of *industry technology* (Model II). Each model, in turn, has two versions: *first*, with competitive imports treated endogenously and, *second*, competitive imports treated exogenously. All time subscripts are omitted in the discussion that follows, to economize on notations.

1. The *Commodity Technology* Model (Model I).

In view of the four points just made and assuming, for the moment, that it is possible to develop a *commodity technology* matrix, we can write the commodity flow balance equations as follows:

$$x_i^D - \sum_{j=1}^{n} (a_{ij}^D + m_{ij})\, x_j^D + g_i\, x_i^D = y_i^D \tag{4.1}$$

$$- g_i\, x_i^D + \underset{\sim}{x}_i^M = 0$$

where

$$x_i^D = \sum_{j=1}^{n} x_{ij}^D + y_i^D$$

is the total domestic output of product or commodity i ($x_i^D = x_j^D$ for $i = j$);

$$a_{ij}^D = x_{ij}^D / x_j^D$$

is the domestic input coefficient (i.e., the amount of domestically produced product i used in the domestic production of one unit of commodity j);

$$m_{ij} = x_{ij}^M / x_j^D$$

is the competitive imports input coefficient, indicating the amount (value) of competitive imports of product i that is required per unit of the domestic production of commodity j;

y_i^D is total final demand for the domestically produced product i;

$$\underset{\sim}{x}_i^M = \sum_{j=1}^{n} x_{ij}^M$$

is the total competitive imports of product i required by the economy for intermediate consumption; and, finally,[1]

$$g_i = \frac{\underset{\sim}{x}_i^M}{x_i^D} = \frac{\sum_{j=1}^{n} x_{ij}^M}{x_i^D} = \frac{\sum_{j=1}^{n} m_{ij}\, x_j^D}{x_i^D}$$

is the total amount (value) of competitive imports of product i that is required by the economy for intermediate consumption per unit of the domestic output of the commodity with which it is in competition.

(4.2)
$$\begin{bmatrix} [I-(A^D + M) + \hat{g}] & 0 \\ -\hat{g} & I \end{bmatrix} \begin{pmatrix} X^D \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{pmatrix} Y^D \\ 0 \end{pmatrix}$$

[1] Obviously, this is a simplification of a functional relationship of the following form:

Let

$$\sum_{j=1}^{n} x_{ij}^M = \underset{\sim}{x}_i^M \quad ;$$

then

$$\underset{\sim}{x}_i^M = \bar{\underset{\sim}{x}}_i^M + g_i x_i^D + u_i$$

where

$\bar{x}_i^M$ is the regression constant assumed to be zero, so that the regression equation is forced through the origin,

g_i is the regression coefficient indicating the marginal propensity of the system to consume competitive imports of product group i (as intermediate goods) per unit of the domestic output of product group i, and

u_i is the disturbance term.

where $\hat{g}$ is a diagonal submatrix consisting of g_i. Finally, in *reduced form,* (4.2) becomes

$$(4.3) \quad \begin{pmatrix} X^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} [I - (A^D + M) + \hat{g}] & 0 \\ \\ -\hat{g} & I \end{bmatrix}^{-1} \begin{pmatrix} Y^D \\ \\ 0 \end{pmatrix} .$$

Consequently, after matrix inversion [refer to Appendix D], we have:

$$(4.4) \quad \begin{pmatrix} X^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} [I - (A^D + M) + \hat{g}]^{-1} & 0 \\ \\ \hat{g} \; [I - (A^D + M) + \hat{g}]^{-1} & I \end{bmatrix}^{-1} \begin{pmatrix} Y^D \\ \\ 0 \end{pmatrix} ,$$

where the upper-left submatrix transforms the vector of final demand for domestically produced products (i.e., Y^D) into domestic output requirements from every sector (i.e., X^D), while the lower-left submatrix helps us determine the competitive import requirements of the system for intermediate consumption in order to produce not only the exogenously specified *bill of goods* Y^D but also the direct and indirect interindustry demand for goods and services set into motion by Y^D. The model is *recursive*, in that the determination of domestic output requirements both precedes and internally generates the requirements for competitive imports.

As it can be easily seen, noncompetitive imports do not enter the model formulation at all (i.e., they are treated exogenously). If the model in its present form is to be used for forecasting, it is necessary to assume that there exists perfect substitution between competitive imports and domestically produced products. Each *combined* input coefficient ($a_{ij}^D + m_{ij}$), denoting total input of type i, regardless of where produced, per unit of the domestic output of product j, is assumed to be constant inpractical applications. An input coefficient in this model may thus be either domestically produced, imported, or a combination of the two, and a substitution of imports for domestic output is assumed to have no effect on the coefficient itself. It is further assumed that the ratio of imports of a competitive commodity required for intermediate consumption *to* the domestic output of the competing domestic commodity, denoted by g_i, is assumed constant (in constant dollars).

This leads directly to the assumption that the foreign and domestic shares of the market for intermediate consumption [$(g_i/1 + g_i)$ and $(1/1 + g_i)$] remain constant. All competitive imports are routed to the intermediate consumers as part of the competing domestic products. That is to say, the production of a domestic commodity, x_i^D, is supplemented by the addition of competitive imports of type i, $\underset{\sim}{x}_i^M$. It is not necessary in this model to specify separately the domestically and foreign-produced components of total final demand (y_i^D and y_i^M), since the model requires the exogenous stipulation of only the final demand for domestically produced products. The model itself allocates total requirements for a given commodity for intermediate consumption into two sources of supply, domestic and foreign, in the same proportions of the total market for intermediate goods as observed during the base year.

The chief weakness of the model as it now stands is that it completely ignores final demand for competitive imports. Consequently, when competitive imports are made endogenous to the model, the model predicts, in addition to domestic output requirements, the economy's competitive import requirements for intermediate consumption only. Another weakness of the model is that competitive imports are assumed to generate no domestic output of services between the time they enter the economy and the time they reach the domestic consumers. This assumption in the model is reflected by the fact that the upper-right submatrices in (4.2), (4.3), and (4.4) are null matrices.

In fact, however, the domestic transportation, wholesale and retail trade, and insurance industries do provide distributive services in connection with competitive imports. As a consequence of this assumption, therefore, the model slightly under-predicts the domestic output levels of the distributive industry sectors.

The shortcoming just described can be overcome by reformulating the model to account for the services rendered by the system before competitive imports reach the domestic consumers. Hence, we can re-write (4.1) as

$$
\begin{aligned}
(4.5) \qquad x_i^D - \sum_{j=1}^{n} (a_{ij}^D + m_{ij})\, x_j^D + g_i\, x_i^D - \sum_{k=1}^{n} h_{ik}\, \underset{\sim}{x}_k^M &= y_i^D \\
- g_i\, x_i^D + \underset{\sim}{x}_i^M &= 0
\end{aligned}
$$

where

$$x_j^D = x_i^D \text{ for } i = j; \quad x_i^M = x_k^M \text{ for } i = k;$$

and

$$(4.6) \qquad {}_r h_i = \frac{\sum_{j=1}^{n} {}_r\Delta x_{ij}^{D(M)}}{\underset{\sim}{x}_i^M} .$$

such that h_{ik} represents total additional or extra domestic output of type i (i.e., transportation and warehousing, wholesale and retail trade, finance and insurance) required per unit of competitive imports of each product (i = 1,2,...,n) used for intermediate consumption.

The derivation of the ${}_r h_i$ coefficients is relatively easy to explain. Assume that associated with every x_{ij}^M (i.e., competitive imports of product i used by sector j as an intermediate good) are three types of distributive services that are rendered domestically by the respective distributive sectors (denoted as ${}_r\Delta x_{ij}^{D(M)}$). Summing across each row (i.e., $\sum_{j=1}^{n} {}_r\Delta x_{ij}^{D(M)}$), we can determine the total value of each of the three types of distributive services (i.e., ${}_r\Delta x_i^{D(M)}$ generated by total competitive imports of product i used by all industries as an intermediate good. It can be seen that for each *margin* service of type r (e.g., transportation) we have a column of h_i (i = 1,2,...,n) coefficients, or r columns of h_i coefficients in total. We can write each such column as a row, using the subscript i for rows and k for columns. We can thus define the nxn matrix $H = [h_{ik}]$, in which all rows except for the r *margin* rows are zero."[2]

Unfortunately, in the 1958 Input-Output Study, each $x_{ri}^{D(M)}$ (i.e., total distributive services of type r rendered in connection with competitive imports of product i used for intermediate consumption) is not separately computed. Each of the three types of distributive services are identified only for all competitive imports taken together (i.e., $\sum_{j=1}^{n} \sum_{i=1}^{n} {}_r\Delta x_{ij}^{D(M)}$). But, somewhat reluctantly, we can still compute each ${}_r h_i$ coefficient as

$$(4.7) \qquad {}_r h_i = \frac{\sum_{j=1}^{n} \sum_{i=1}^{n} {}_r\Delta x_{ij}^{D(M)}}{\sum_{i=1}^{n} \underset{\sim}{x}_i^M}$$

[2] In these three rows, entries under the three (or more) columns corresponding to the distributive sectors in question will likewise be zero.

Where each ${}_r h$ can be given the same interpretation as ${}_r h_i$. For each r or *margin* service, only one such coefficient can be computed. When we assume that the marginal propensity of the system to generate distributive services of type r per unit of competitive imports used for intermediate consumption is the same, regardless of the particular product that is imported, then the elements in a given *margin* row in the H matrix all have the same numerical value.

In this way, we can re-write the model expressed in Eqs. (4.2) and (4.3) as

$$(4.8)\qquad \begin{bmatrix} [I - (A^D + M) + \hat{g}] & -H \\ -\hat{g} & I \end{bmatrix} \begin{pmatrix} X^D \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{pmatrix} Y^D \\ 0 \end{pmatrix}$$

which can be written in *reduced form* as

$$(4.9)\qquad \begin{pmatrix} X^D \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} [I - (A^D + M) + \hat{g}] & -H \\ -\hat{g} & I \end{bmatrix}^{-1} \begin{pmatrix} Y^D \\ 0 \end{pmatrix}.$$

After inverting the parameter matrix [refer to Appendix D] we finally have:

(4.10)

$$\begin{pmatrix} X^D \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} \left\{[I - (A^D + M) + \hat{g}] - H\hat{g}\right\}^{-1} & \left\{[I - (A^D + M) + \hat{g}] - H\hat{g}\right\}^{-1} H \\ \hat{g}\left\{[I - (A^D + M) + \hat{g}] - H\hat{g}\right\}^{-1} & I + \hat{g}\left\{[I - (A^D + M) + \hat{g}] - H\hat{g}\right\}^{-1} H \end{bmatrix} \begin{pmatrix} Y^D \\ 0 \end{pmatrix}$$

Here, the *upper-left submatrix* transforms the *bill of goods* for domestically produced products into total domestic output requirements from each sector, by specifying the total (direct and indirect) domestic output required from each sector per unit of final demand for domestically produced products. The *upper-right submatrix* helps us determine how much of a decrease in the domestic production of each commodity will result from a shift in final demand from domestic products to competitive imports, under the *perfect substitutability* assumption and subject to the assumptions inherent in $\hat{g}$ and H whose numerical values are fixed at the base year (e.g., 1958) values. The *lower-left submatrix* numerically transforms final demand for domestically produced products into total requirements for competitive imports of each commodity, under the same set of assumptions. The *lower-right submatrix*, finally, helps us determine the reduction that will result in the total competitive imports requirements of the economy for each commodity *from* a shift in final demand from domestic products to competitive imports,[3] again under the same set of assumptions. As before, Eq. (4.10) represents a *recursive* system.

The chief weakness of this model, which is a formal statement of the situation typified by the 1958 Input-Output Study in the United States, is that no relationship is assumed to exist between final demand for competitive imports and the performance of the domestic economy. Some of these reverse or *feedback* relationships will be spelled out in extensions of this model given in Chapter V. In retrospect, then, it appears that suppressing the Y^M vector into a single total[4] in the 1958 Input-Output Study was an unfortunate step from an analytical standpoint. It is only to be hoped that the situation will be rectified in the 1963 Study which is currently underway. As perhaps a relatively minor point, it is also hoped that the value of each type of domestic distributive service that is generated by competitive imports of each product will be separately identified, so that the submatrix H can be computed and used with ease.

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[3] It should be understood that an increase in final demand for competitive imports is assumed here to represent a shift in final demand from domestically produced products to competitive imports.

[4] More accurately, several totals are given, each showing the total purchases of competitive imports by a particular final demand sector.

When competitive imports required for intermediate consumption are assumed to be exogenously determined, the model, of course, takes a completely different form. Instead of Eq. (4.8) we then have

$$(4.11)\quad \begin{bmatrix} [I-(A^D+M)]^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} X^D \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{pmatrix} Y^D - \underset{\sim}{X}^M \\ \underset{\sim}{X}^M \end{pmatrix},$$

and Eq. (4.10) now becomes

$$(4.12)\quad \begin{pmatrix} X^D \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} [I-(A^D+M)]^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} Y^D - \underset{\sim}{X}^M \\ \underset{\sim}{X}^M \end{pmatrix}.$$

To summarize, while the *reduced form* statement of the model given in Eq. (4.10) treats competitive imports as endogenous, the model formulation given in Eq. (4.12) treats competitive imports as exogenous. It is assumed, in both cases, that the *technological* coefficients matrix (A^D + M) represents a *commodity technology* matrix. In fact, the model discussed so far will be called the *commodity technology* model (or Model I), which has two versions: the *first* is the formulation under Condition A, when competitive imports are endogenous as in Eq. (4.10), and the *second* is the formulation under Condition B, when competitive imports are exogenous as in Eq. (4.12).

2. The *Industry Technology* Model (Model II)

As mentioned earlier quite frequently, the *technological* coefficients matrix is more readily observable as an *industry technology* matrix, since the basic intersectoral flows information used in input-output model construction is compiled in most countries in *product-to-industry flows* terms. To repeat, the *commodity technology* matrix is a *derived* matrix, computed directly from the *product-to-industry flows* data, by making certain assumptions, as indicated in Chapter II. Since the *commodity technology* matrix is in most instances only a *derived* matrix and not a directly observed matrix, the quality of the resulting *commodity technology* model is difficult to ascertain, except in a relative sense when the predictive results are compared with those obtained from an *industry technology* model.

Assuming *industry technology*, we can write the commodity flow balance equations in (4.5) as follows:

$$(4.13)\qquad \begin{aligned} x_i^D &= \sum_{j=1}^{n} (a_{ij}^D + m_{ij})\,\tilde{x}_j^D - g_i\,\tilde{x}_j^D + \sum_{k=1}^{n} h_{ik}\,\underset{\sim}{x}_k^M + y_i^D \\ \underset{\sim}{x}_i^M &= \qquad\qquad\qquad\qquad + g_i\,\tilde{x}_j^D \end{aligned}$$

where

x_i^D, $\underset{\sim}{x}_i^M$, y_i^D, and h_{ik} are the same as before;

$x_i^D \neq \tilde{x}_j^D$ for i = j, since x_i^D refers to total domestic output of product i (wherever produced, that is, regardless of which *industry* produces it), whereas $\tilde{x}_j^D$ represents the total domestic output of *industry* j (i.e., *industry* j's primary and secondary products);

$a_{ij}^D = x_{ij}^D / \tilde{x}_j^D$ is the domestic *industry* input-coefficient;

$(a_{ij}^D + m_{ij})$ is the *combined* matrix of *technological* coefficients, representing *industry technology*;

$$g_i = \frac{\underset{\sim}{x}_i^M}{\tilde{x}_i^D} = \frac{\sum_{j=1}^{n} x_{ij}^M}{\tilde{x}_i^D} = \frac{\sum_{j=1}^{n} m_{ij}\,\tilde{x}_j^D}{\tilde{x}_i^D}$$

is the marginal (and average) propensity of the economy to consume competitive imports of product i (as intermediate goods) per unit of the domestic output of *industry* j.[5]

[5] It should be noted that the coefficients g_i, as defined here, are somewhat different from those given earlier in Eq. (4.1). Here domestic *industry* output levels $\tilde{x}_i^D$ are used as the independent variables, as opposed to domestic *product* output levels x_i^D that were used as the independent variables in Eq. (4.1).

Re-writing Eq. (4.13) more compactly we have:

$$(4.14)\qquad \begin{pmatrix} X^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} [(A^D + M) - \hat{g}] & H \\ \\ \hat{g} & 0 \end{bmatrix} \begin{pmatrix} \tilde{X}^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} + \begin{pmatrix} Y^D \\ \\ 0 \end{pmatrix}$$

where competitive imports required for final consumption are left out for the same reasons as before.

In order to make this model operational, it is necessary to eliminate the vector $\tilde{X}^D$ (i.e., domestic *industry* output levels) from the system of equations by using a transformation mechanism which expresses the vector $\tilde{X}^D$ as a linear function of the vector X^D (i.e., total domestic *product* output levels). This transformation mechanism has already been spelled out in Section F of Chapter II. Namely, two new matrices are introduced: the *make* matrix and the *industry product mix* matrix, with the latter derived from the former. It will be recalled that the *make* matrix is defined as $\dot{X} = [\dot{x}_{kl}]$, in which a given element $\dot{x}_{kl}$ expresses the total amount (value) of product l domestically produced by industry k. Hence, total domestic output of product l can be computed as $\dot{x}_k = \sum_{k=1}^{n} \dot{x}_{kl}$ (i.e., summing over all rows for each column or product category). The *industry product mix* matrix $U = [u_{kl}]$ can then be computed as $U = [\dot{x}_{kl}]\ \hat{\dot{x}}^{-1}$, where $\hat{\dot{x}}$ is a diagonal matrix consisting of $\dot{x}_{kl}$ (i.e., total domestic *product* output levels) and $\hat{\dot{x}}^{-1}$ is its inverse. More simply, each element u_{kl} in matrix U is defined as

$$(4.15)\qquad u_{kl} = \frac{x_{kl}}{\sum_{k=1}^{n} \dot{x}_{kl}} \quad \circ \quad k, l, = 1,2,\ldots,n$$

It can be seen that each $\dot{x}_k = \sum_{k=1}^{n} \dot{x}_{kl}$ refers to the same variable as represented by the standard notation x_i^D.

Using the matrix $U = [u_{kl}]$, we can express the domestic *industry* output levels as a linear combination of domestic *product* output levels as follows:

$$(4.16) \qquad \tilde{X}^D = UX^D,$$

where the matrix U apportions the total domestic output of each product into the respective industries producing it, so that each row sum

$$\tilde{x}_k^D = u_{k1}\, x_1^D + u_{k2}\, x_2^D + \ldots + u_{kn}\, x_n^D \qquad (k = 1,2,\ldots,n)$$

yields the total domestic output of each *industry*.

Substituting (4.16) into (4.14), we obtain

$$(4.17) \qquad \begin{pmatrix} X^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} [(A^D + M) - \hat{g}] & H \\ \\ \hat{g} & 0 \end{bmatrix} \begin{pmatrix} UX^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} + \begin{pmatrix} Y^D \\ \\ 0 \end{pmatrix}$$

which is the same as

$$(4.18) \qquad \begin{pmatrix} X^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} [(A^D + M) - \hat{g}]\, U & H \\ \\ \hat{g}\, U & 0 \end{bmatrix} \begin{pmatrix} X^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} + \begin{pmatrix} Y^D \\ \\ 0 \end{pmatrix}.$$

After transferring terms to the left side, (4.18) becomes

$$(4.19) \qquad \begin{pmatrix} X^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} - \begin{bmatrix} [(A^D + M) - \hat{g}]\, U & H \\ \\ \hat{g}\, U & 0 \end{bmatrix} \begin{pmatrix} X^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{pmatrix} Y^D \\ \\ 0 \end{pmatrix}.$$

When all the multiplications are carried out, (4.19) results in

$$X^D - [(A^D + M) - \hat{g}]\, UX^D - H\underset{\sim}{X}^M = Y^D \tag{4.20}$$

$$\underset{\sim}{X}^M - \hat{g}\, UX^D \qquad\qquad - 0 \cdot \underset{\sim}{X}^M = 0$$

where the lower equation can be reorganized as

$$- \hat{g}\, UX^D + \underset{\sim}{X}^M - 0 \cdot \underset{\sim}{X}^M \qquad = 0. \tag{4.21}$$

By factoring out the vectors X^D and $\underset{\sim}{X}^M$, we can re-write (4.20) as

$$\begin{bmatrix} [I - (A^D + M)\, U + \hat{g}\, U] & -H \\ -\hat{g}\, U & I \end{bmatrix} \begin{pmatrix} X^D \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{pmatrix} Y^D \\ 0 \end{pmatrix}. \tag{4.22}$$

Finally, the *industry technology* model can be written in *reduced form* as

$$\begin{pmatrix} X^D \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} [I - (A^D + M)\, U + \hat{g}\, U] & -H \\ -\hat{g}\, U & I \end{bmatrix} \begin{pmatrix} Y^D \\ 0 \end{pmatrix}. \tag{4.23}$$

After inverting the partitioned matrix in (4.23) we obtain the following submatrices [refer to Appendix D] : the *upper-left submatrix*

$$\left\{ [I - (A^D + M)\, U + \hat{g}\, U] - H\, \hat{g}\, U \right\}^{-1} \quad ; \tag{4.24a}$$

the *upper-right submatrix*

$$\left\{ [I - (A^D + M)\, U + \hat{g}\, U] - H\, \hat{g}\, U \right\}^{-1} H \quad ; \tag{4.24b}$$

the *lower-left submatrix*

$$(4.24c)\qquad \hat{g}U \quad \left\{[I-(A^D+M)U+\hat{g}U] \; - \; H\hat{g}U\right\}^{-1} \quad ;$$

and the *lower-right submatrix*

$$(4.24d)\qquad I + \hat{g}U \quad \left\{[I-(A^D+M)U+\hat{g}U] \; - \; H\hat{g}U\right\}^{-1} H \quad .$$

The *upper-right* and *lower-right submatrices* are not really required by this model, since they fall out after being post-multiplied by a null vector. The *upper-left submatrix,* when post-multiplied by the exogenous vector of final demand for domestically produced products, helps us determine the total domestic output requirements for each product. The *lower-left submatrix* transforms the vector of final demand for domestically produced products into competitive imports requirements of the economy for intermediate consumption.

The four submatrices in (4.24a) through (4.24d) represent the parameters of the *industry technology* model (Model II), when competitive imports required by the economy for intermediate consumption are made endogenous to the model. A comparison of these four submatrices with those given in Eq. (4.10) should show the explicit mathematical, as well as empirical, differences between the two models, under Condition A (i.e., when competitive imports are endogenous).

When competitive imports are made exogenous to the model, the structure of the model becomes obviously quite different. Instead of (4.14), we now have

$$(4.25)\qquad \begin{pmatrix} X^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} (A^D+M) & 0 \\ \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \tilde{X}^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} + \begin{pmatrix} Y^D - \underset{\sim}{X}^M \\ \\ \underset{\sim}{X}^M \end{pmatrix}$$

where the exogenous variable is now $Y^D - \underset{\sim}{X}^M$, that is, the difference between the final demand for domestically produced products and the competitive imports required for intermediate consumption. As before, final demand for competitive imports are completely ignored.

When we substitute (4.16) into (4.25) and go through the same type of mathematical manipulation as was done in (4.17) through (4.22), we end up with

$$(4.26)\qquad \begin{pmatrix} X^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} [I-(A^D+M)\,U & 0 \\ \\ 0 & I \end{bmatrix}^{-1} \begin{pmatrix} Y^D - \underset{\sim}{X}^M \\ \\ \underset{\sim}{X}^M \end{pmatrix}.$$

Finally, after inverting the parameter matrix, we obtain

$$(4.27)\qquad \begin{pmatrix} X^D \\ \\ \underset{\sim}{X}^M \end{pmatrix} = \begin{bmatrix} [I-(A^D+M)\,U]^{-1} & 0 \\ \\ 0 & I \end{bmatrix} \begin{pmatrix} Y^D - \underset{\sim}{X}^M \\ \\ \underset{\sim}{X}^M \end{pmatrix}.$$

Post-multiplying the *upper-left submatrix* by the vector $Y^D - \underset{\sim}{X}^M$ (i.e., the difference between the final demand vector for domestically produced products *and* the vector of competitive imports required for intermediate consumption) yields the domestic output requirements for each product. The structural differences between the *industry technology* model *and* the *commodity technology* model, when competitive imports are made exogenous to the model, can be seen by comparing the systems expressed in (4.12) and (4.27).

3. Summary Statement On the Models

Two alternative models have been developed for use in the experiments reported in this chapter, the *first* based on the hypothesis of *commodity technology*(Model I) and the *second* based on the hypothesis of *industry technology* (Model II). Further, these two models are formulated under two different conditions: *first* under the condition that competitive imports are endogenous to the model (Condition A) and, *second*, under the condition that they are exogenous to the model (Condition B).

The *matrix multipliers* corresponding to Model I – Condition A are given in Eq. (4.10), while those corresponding to Model II – Condition A are given in (4.24a) through (4.24d). In addition, the *matrix multipliers* corresponding to Model I – Condition B are given in Eq. (4.12), while those corresponding to Model II – Condition B are given in Eq. (4.27).

For the purpose of the experiments reported in this chapter, only the *exogenous* versions of the *commodity technology* and *industry technology* models, given in Eqs. (4.12) and (4.27) have been used. The *matrix multipliers* or the Leontief inverse matrix in each of these two *exogenous* model versions have been empirically derived at four different levels of sectoral aggregation (i.e., 79 x 79, 60 x 60, 45 x 45, and 17 x 17). All the parameters of the *endogenous* model versions have also been empirically estimated, but the inverse matrices have not been computed. In one or two tries, certain elements in the Leontief inverse were observed to be negative, and the elements in two separate rows all turned out to be negative, leading to negative intermediate demand and output predictions. The reasons for these results were not readily apparent, and because of severe time constraints, attention in the experiments were focused entirely on the *exogenous* model versions. The experiments reported in this chapter can be extended in the future, using the two *endogenous* model versions in making conditional point predictions of intermediate demand and domestic output levels and competitive import requirements.

C. THE STRUCTURE OF INPUT-OUTPUT PREDICTION ERRORS

The effect of measurement errors on input-output predictions were summarized in Eq. (1.30) as follows:

$$(4.28) \qquad X^p_{t+\tau} - \bar{X}^p_{t+\tau} = [(I - \bar{A}_t)^{-1} + (I - \bar{A}_t)^{-1} \tilde{A}_t (I - \bar{A}_t)^{-1} \tilde{Y}_{t+\tau} + (I - \bar{A}_t)^{-1} \tilde{A}_t (I - \bar{A}_t) \bar{Y}_{t+\tau}$$

where

$X^p_{t+\tau}$ is the *predicted* output vector for year $t + \tau$;

$\bar{X}^p_{t+\tau}$ represents the *true* prediction of the output vector τ years into the future using a *technology* matrix for year t;

$A_t = (\bar{A}_t + \tilde{A}_t)$ is the observed *technology* matrix for year t, which is decomposed into the *true* matrix $\bar{A}_t$ and the error matrix $\tilde{A}_t$;

$Y_{t+\tau} = (\bar{Y}_{t+\tau} + \tilde{Y}_{t+\tau})$ is the observed final demand vector for the prediction year, decomposed into the *true* vector $\bar{Y}_{t+\tau}$ and the error vector $\tilde{Y}_{t+\tau}$.

Suppose the observed output vector for year $t + \tau$ was $X_{t+\tau} = (\bar{X}_{t+\tau} + \tilde{X}_{t+\tau})$, similarly decomposed into a *true* vector $\bar{X}_{t+\tau}$ and an error vector $\tilde{X}_{t+\tau}$. Then, combining this with the results of Eqs. (1.29) and (1.30), the input-output prediction error can be written as

$$(4.29) \qquad X^p_{t+\tau} - X_{t+\tau} = [(I - \bar{A}_t)^{-1} \bar{Y}_{t+\tau} + (I - \bar{A}_t)^{-1} \tilde{Y}_{t+\tau} + (I - \bar{A}_t)^{-1} \tilde{A}_t (I - \bar{A}_t)^{-1} (\bar{Y}_{t+\tau} + \tilde{Y}_{t+\tau})] - (\bar{X}_{t+\tau} + \tilde{X}_{t+\tau})$$

which can be decomposed into a *true* prediction error vector and a measurement error vector. Clearly, the measurement error vector consists of the right-hand side of Eq. (4.29) *minus* $\tilde{X}_{t+\tau}$. The *true* prediction error then consists of the right-hand side of Eq. (4.29) *minus* the measurement error vector as just defined.

It follows directly from these results that at the level of individual input-output sectors, the prediction error $e_{i,t+\tau}$ can be written as

$$(4.30) \qquad e_{i,t+\tau} = \bar{e}_{i,t+\tau} + \tilde{e}_{i,t+\tau}$$

where $\bar{e}_{i,t+\tau}$ is the *true* prediction error for sector i (i=1,...,n), which would be experienced if no observational errors were committed, and where $\tilde{e}_{i,t+\tau}$ is the observational or measurement error component of the prediction error for sector i.

Following Theil,[6] given a series of *technology* matrices for a consecutive number of years so that each can be used to make conditional point predictions for the subsequent year, the true prediction errors $\bar{e}_{i,t+1}$ one year ahead (which Theil calls elementary prediction errors) can be assumed to be random variables with mean μ_i and variance σ_i^2 (independent of time) and that they are uncorrelated over time. Also, it can be assumed that $\mu_i = 0$ holds for a large number of sectors. Some of these assumptions can be tested by using the observed elementary prediction errors $e_{i,t+1}$, provided that additional assumptions about the measurement-error components of the elementary prediction errors $\tilde{e}_{i,t+1}$ are made.[7] The assumption that $\mu_i = 0$ can be tested, using a t-test, provided that an *ad hoc* assumption is made that the distribution of $e_{i,t+1}$ is normal (with mean μ_i and variance σ_i^2 by approximation.[8]

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[6] Henri Theil, *Applied Economic Forecasting* (Amsterdam: North-Holland Publishing Co., and Chicago: Rand McNally and Co., 1966), p. 215.

[7] It is necessary to assume that $\tilde{e}_{i,t+1}$ has a negligible expected value and a variance, var $\tilde{e}_i$, which is independent of the time span (and of the base year). A further assumption must be made that $\tilde{e}_i$ is uncorrelated with $\tilde{e}_{i,t+1}$. A justification for these assumptions can be found in C.B. Tilanus, *Input-Output Experiments, The Netherlands, 1948-1961* (Rotterdam: Rotterdam University Press, 1966), p. 110.

[8] If the hypothesis $\mu_i = 0$ is true, then the ratio

$$t_i = \frac{\frac{1}{s}\sum_{t=1}^{s} e_{i,t+1}}{\sqrt{m_{i1}}} \cdot \sqrt{s}$$

has a t-distribution with s degrees of freedom, where m_{i1} is the mean square prediction error one year ahead

$$m_{i1} = \frac{1}{s}\sum_{t=1}^{s} e_{i,t+1}^2 \qquad (t = 1,\dots,s).$$

Theil defines $e_{i,t+1}$ as (using his own notations)

$$e_{it\tau} = \log \frac{z_{it\tau}^p}{z_{i,t+\tau}}$$

where the numerator denotes prediction and the denominator denotes the realization (See Theil, *op. cit.*, p. 214).

On the other hand, Tilanus defines $e_{i,t+1}$ as (using his own notations)

$$e_{its} = \log z_{its}^p - \log z_i\ (t+s)$$

where z_{its}^p is the predicted value and z_i (t + s) is the realized value (intermediate demand) in the prediction year (t + s). See Tilanus, *ibid.*, p. 55. In both Theil and Tilanus, logarithms refer to natural logarithms.

We can pursue the theoretical analysis of input-output prediction errors at the sectoral level by considering the logarithmic prediction errors.[9] First, we can re-write Eq. (4.30) as follows

$$(4.31) \qquad e_{i,t+\tau} = \log \frac{z^p_{i,t+\tau}}{z_{i,t+\tau}} = \log \left(\frac{\bar{z}^p_{i,t+\tau}}{\bar{z}_{i,t+\tau}} \circ \frac{z^p_{i,t+\tau}/\bar{z}^p_{i,t+\tau}}{z_{i,t+\tau}/\bar{z}_{i,t+\tau}} \right)$$

where

$$\bar{e}_{i,t+\tau} = \log \frac{\bar{z}^p_{i,t+\tau}}{\bar{z}_{i,t+\tau}}$$

$$\tilde{e}_{i,t+\tau} = \log \frac{z^p_{i,t+\tau}}{\bar{z}^p_{i,t+\tau}} - \log \frac{z_{i,t+\tau}}{\bar{z}_{i,t+\tau}}$$

and where the superscript p denotes prediction, the bar ("–") indicates *true* values, and the absence of p or ("–") indicates actual observations. It can be seen that the observational or measurement error component $\tilde{e}_{i,t+\tau}$ consists of two separate terms, one which refers to the measurement error of prediction and the other represents the measurement error of the realization.[10]

Theil and Tilanus have jointly developed what they have called the *cumulation rule*, which states that a prediction error for a time span of τ years between the prediction year $t + \tau$ and the base year t is, with great precision, equal to the sum of all τ elementary

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[9] The reasons for using logarithmic prediction errors will be explained later.

[10] Note that both the *true* and the observed prediction errors are identically zero for $\tau = 0$:

$$e_{i,t+0} \equiv 0, \quad \bar{e}_{i,t+0} \equiv 0$$

which is due to the fact that $z^p_{i,t+0} \equiv z_{i,t}$, $\bar{z}^p_{i,t+0} \equiv \bar{z}_{i,t}$, implying that the two terms of $\tilde{e}_{i,t+0}$ cancel against each other:

$$\log \frac{z^p_{i,t+0}}{\bar{z}^p_{i,t+0}} \equiv \log \frac{z_{i,t}}{\bar{z}_{i,t}} .$$

See Theil, *op, cit.,* p. 215.

year-to-year) prediction errors for the year ahead, following the base year and inclusive of the prediction year:[11]

(4.32) $$e_{i,t+\tau} = \sum_{s=t}^{t+\tau-1} e_{i,s+1} + d_{i,t+\tau}$$

where, $d_{i,t+\tau}$, the discrepancy from the cumulation rule, are small and are assumed to be uncorrelated with the elementary prediction errors.

It will be first shown that the cumulation rule holds for the true prediction errors

(4.33) $$\bar{e}_{i,t+\tau} = \sum_{s=t}^{t+\tau-1} \bar{e}_{i,s+1} + \bar{d}_{i,t+\tau}$$

where the *true* discrepancy from the cumulation rule $\bar{d}_{i,t+\tau}$

(4.34) $$\bar{d}_{i,t+\tau} = d_{i,t+\tau} - \tilde{d}_{i,t+\tau}$$

is close to zero. Under the assumptions that (a) the true prediction errors $\bar{e}_{i,t+1}$ one year ahead are random variables with mean μ_i and variance σ_i^2 (independent of t) and that they are uncorrelated over time, and (b) that $\mu_i = 0$ holds for most of the sectors, we can express the expected value of the true prediction error τ years ahead as

(4.35) $$E(\bar{e}_{i,t+\tau}) = \tau \mu_i$$

where the expectation of the *true* discrepancy $\bar{d}_{i,t+\tau}$ from the cumulation rule is neglected, since the discrepancies are very small and of varying signs.[12] The variance of $\bar{e}_{i,t+\tau}$ can be found as

(4.36) $$\text{var } \bar{e}_{i,t+\tau} = \tau \sigma_i^2 + \text{var } \bar{d}_{i,t+\tau}$$

[11] For a detailed discussion and mathematical derivation of the cumulation rule, refer to Theil, *ibid.*, pp. 239-248, and Tilanus, *op. cit.*, pp. 95-101. For the derivation of the cumulation rule, two basic assumptions are made: first, that the elements of the final demand vector move approximately proportionately over time, and second, that the elements of the matrix multiplier are approximately constant.

[12] Theil, *op. cit.*, p. 219.

under the assumption that the $\bar{d}$'s are uncorrelated with the *true* elementary prediction errors. The mean square can be found by adding the squared mean to the variance:

$$(4.37) \qquad E(\bar{e}^2_{i,t+\tau}) = \tau\,\sigma_i^2 + \text{var}\,\bar{d}_{i,t+\tau} + \tau^2\,\mu_i^2$$

The first term on the right increases proportionately with time, τ. The last term increases more than proportionately with time, except when $\mu_i = 0$, in which case it vanishes. As to the middle term, Theil reports that the mean square discrepancy from the cumulation rule in observed errors increases rather substantially relative to the mean square of $e_{i,t+\tau}$, which itself increases less than proportionally with τ.[13] Theil suggests that var $\bar{d}_{i,t+\tau}$ increases at least proportionally with τ, presumably more than proportionally.[14] In conclusion, when the mean square of the true error $\bar{e}_{i,t+\tau}$ does *not* increase proportionally with τ, it will increase *more* than proportionally.[15]

Using the cumulation rule and the concept of measurement errors, the prediction error $e_{i,t+\tau}$ can be decomposed as follows

$$(4.38) \qquad e_{i,t+\tau} = \tilde{e}_{i,t+\tau} + \sum_{s=t}^{t+\tau-1} \bar{e}_{i,s+1} + \bar{d}_{i,t+\tau}\ ,$$

where the measuremen error component $\tilde{e}_{i,t+\tau}$ consists of two parts, one of which is the logarithmic measurement error of the forecast $z^p_{i,t+\tau}$ and the other the logarithmic measurement error of the realization, $z_{i,t+\tau}$. On the assumption that these two measurement errors are random variables with zero mean, we find that the expected value of the observed $e_{i,t+\tau}$ is the same as that of $\bar{e}_{i,t+\tau}$:

$$(4.39) \qquad E(e_{i,t+\tau}) = \tau\,\mu_i\ .$$

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[13] *Ibid.*

[14] *Ibid.*

[15] *Ibid.*

When we further assume that the measurement errors are uncorrelated with the true errors, we have the following variance:

$$\text{var } e_{i,t+\tau} = \text{var } \tilde{e}_{i,t+\tau} + \tau \sigma_i^2 + \text{var } \bar{d}_{i,t+\tau} \tag{4.40}$$

and the following mean square:

$$E(e^2_{i,t+\tau}) = \text{var } \tilde{e}_{i,t+\tau} + \tau \sigma_i^2 + \text{var } \bar{d}_{i,t+\tau} + \tau^2 \mu_i^2 . \tag{4.41}$$

The variance of $\tilde{e}_{i,t+\tau}$ can be written as the sum of two variances minus a double covariance:

$$\text{var } \tilde{e}_{i,t+\tau} = \text{var}\left(\log \frac{z^p_{i,t+\tau}}{\bar{z}^p_{i,t+\tau}} \right) + \text{var}\left(\log \frac{z_{i,t+\tau}}{\bar{z}_{i,t+\tau}} \right) \tag{4.42}$$

$$- 2 \text{ cov}\left(\log \frac{z^p_{i,t+\tau}}{\bar{z}^p_{i,t+\tau}} , \log \frac{z_{i,t+\tau}}{\bar{z}_{i,t+\tau}} \right).$$

The first right-hand variance can be assumed to be independent of τ, since there is no reason to expect that the relative measurement errors of the prediction $z^p_{i,t+\tau}$ varies systematically with τ. The same assumption can be made on the second. Similarly, there is no reason to suppose that the covariance will vary systematically with τ. It probably will not vanish, since the measurement error of $z^p_{i,t+\tau}$ involves $\tilde{y}_{t+\tau}$ which is presumably statistically related to $\tilde{z}_{t+\tau}$. It is thus reasonable to assume that the variance of $\tilde{e}_{i,t+\tau}$ is independent of τ. If an additional assumption is made that there is no dependence on t either, var $\tilde{e}_{i,t+\tau}$ becomes a function of the sector index i only.

In conclusion, the mean square of the observed prediction error consists of three parts, one of which (var $\tilde{e}_{i,t+\tau}$) is a constant for each sector i, the second ($\tau \sigma_i^2$) increases proportionally with τ, and the third (var $\bar{d}_{i,t+\tau} + \tau^2 \mu_i^2$) more than proportionally. Theil found that, on the whole, the mean square error varies less than proportionally with τ.[16]

[16] *Ibid.*, p.220. This finding is based on the small value of the third part, which itself increases more than proportionally (pp. 220-221).

So far, the discussion has centered around the structure of input-output prediction errors at a given level of sectoral aggregation. The next question of interest is as follows. Suppose that the input-output system available at a given level of sectoral aggregation is further condensed into smaller (i.e., more aggregated) systems, using the same basic information available for the most detailed system that happens to be available. Suppose, further, that the series of more and more aggregated input-output systems thus generated are used respectively in making conditional point predictions of intermediate demand and output levels in a given year in the future, and that measures of overall predictive performance are computed. The question we would then like to ask is as follows: if overall measures of predictive performance at different aggregation levels can be considered as random variables, are they statistically independent of one another?

We can perhaps attempt to tackle this question by first reviewing the nature of the *aggregation bias* in input-output predictions, discussed in detail in Chapter I. It will be recalled that we have, to begin with, a detailed matrix A, of order nxn. After aggregation, A is replaced by the matrix EAV′, where EAV′ is the mxm aggregated *technology* matrix. The detailed vectors $X = (x_i)$ and $Y = (y_i)$ are similarly replaced by EX and EY. Predictions of aggregated output levels would then be given by

(4.43) $$EX_{t+\tau}\ (I - EA_t V')^{-1}\ E\,Y_{t+\tau}$$

where

$EX_{t+\tau}$ is the aggregated output prediction;

$EY_{t+\tau}$ is the aggregated final demand vector in year $t + \tau$ that is exogenously determined;

E and V are respectively the matrices showing the aggregation procedure and the weights to be used (refer to a full discussion in Chapter I); and,

A is the *technology* matrix that is used for further aggregation. It should be noted that A is a somewhat more complicated term if the original input-output model is formulated as an *industry technology* model.

The *aggregation bias* was mathematically expressed in Eq. (1.41) as

(4.44) $$G = (I - EAV')^{-1}\,E - E\,(I - A)^{-1},$$

so that the error introduced into the conditional point prediction of aggregated output (or intermediate demand) levels by the aggregation process can be shown to be

$$(4.45) \qquad GY_{t+\tau} = (I - E\,A_t V')^{-1}\, EY_{t+\tau} - E(I - A_t)^{-1}\; Y_{t+\tau}$$

which is the same as Eq. (1.40) except for the time subscripts. The term $GY_{t+\tau}$, then, is the *aggregation bias* component of the prediction error vector.

The prediction error vector can thus be decomposed into the *measurement error, pure prediction error,* and *aggregation bias* components. Let the prediction error vector associated with a detailed model (e.g., 79 x 79) be represented by $\Delta X^p_{t+\tau} = (e_i)$ and the prediction error vector associated with an aggregated model (e.g., 60 x 60) be represented by $E\Delta X^p_{t+\tau} = (e^*_k)$, where i = 1,...,n and k = 1,...,m. It is necessary to explore the stochastic independence of these two vectors.

First, these two prediction error vectors can be expressed in a number of ways, as will be seen a few pages ahead, for the purpose of developing measures of overall predictive performance expressed as scalars. Suppose that these two prediction error vectors are reduced to two indices of overall predictive performance, where each of the two indices can be called *weighted mean logarithmic prediction error,* denoted as θ^*_1 (referring to the detailed model, e.g., 79 x 79) and θ^*_2 (referring to the aggregated model, e.g., 60 x 60), to be explained later in greater detail. We have thus reduced the question to the stochastic independence of θ^*_1 and θ^*_2, taken as two random variables *summarizing* the two prediction error vectors.

In order for the two random variables θ^*_1 and θ^*_2 to be independent, it is necessary and sufficient that the distribution function of the system (θ^*_1, θ^*_2) be equal to the product of the distribution functions of their components:[17]

$$(4.46) \qquad F(\theta_1^*, \theta_2^*) = F_1(\theta_2^*) \times F_2(\theta_2^*)$$

To describe a system of two random variables, we may use besides the mathematical expectations and variances of the components, certain other characteristics, such as the covariance and the correlation coefficient.

The covariance of two random variables θ_1^* and θ_2^* is defined as the expected value of the product of the deviations of θ_1^* and θ_2^* from their expected values:

$$(4.47) \qquad \text{Cov}\,[\theta_1^*, \theta_2^*] = E[(\theta_1^* - E[\theta_1^*])(\theta_2^* - E[\theta_2^*])] \ .$$

The covariance shows the relationship between the variables θ_1^* and θ_2^*. The covariance is equal to zero if θ_1^* and θ_2^* are independent.[18] Consequently, if their covariance (or, what amounts to the same thing, their correlation coefficient) is different from zero, θ_1^* and θ_2^* are said to be correlated. Two correlated variable are also dependent, thus, correlation

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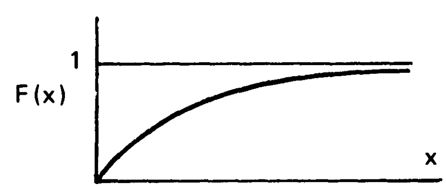
[17] The function F(x), which, for every value of X, gives the probability that the random variable θ^* will assume a value less than x, that is, the function

$$F(x) = P(\theta^* < x)$$

is called the distribution function, defined as:



where F(x) is the probability that the random variable will assume a value that is represented on the real axis by a point lying to the left of the point x, and x is a real number. See, for example, V.E. Gmurman, *Fundamentals of Probability Theory and Mathematical Statistics,* Translated by Scripta Technica, Ltd. (London: Iliffe Books, Ltd., and New York: American Elsevier Publishing Co., Inc., 1968), pp. 108-109.

[18] For a proof of this theorem, see *Ibid.,* pp. 165-166. For a general discussion of the topics mentioned here, refer to K.A. Brownlee, *Statistical Theory and Methodology in Science and Engineering,* Second Edition (New York: John Wiley and Sons, Inc., 1965), pp. 77-80.

between two random variables implies their dependence but their dependence does not imply correlation between them. Conversely, independence of two random variables implies zero correlation, but zero correlation between two random variables does not imply their independence. However, it should be noted that zero correlation between normally distributed variables does imply their independence.[19]

There are a number of tests available for testing for the independence of a sequence of observations, such as the mean square successive difference test, the runs test, the test for serial correlation, etc.[20] In economic analysis, successive observations are not independent in time series analysis and in many other contexts, so that using a model for hypothesis testing which requires (specifies) that the observations are drawn randomly and independently from some population is inappropriate. On the other hand, the available methods for testing the hypothesis that a sample is random are not altogether satisfactory, except when the observations have a meaningful time-order or *sequence*, in which cases it may be possible to detect certain types of deviations from randomness.[21] In the experiments reported in this chapter, the observations θ_1^*, θ_2^*, etc., have been tested for randomness by using the mean square successive difference test. These observations do show a definite sequence, as they measure the overall predictive performance of *commodity technology* and *industry technology* models at different levels of sectoral aggregation.

[19] Gmurman, *op. cit.,* pp. 166-168.

[20] For a discussion of these tests, refer to Brownlee, *op. cit.,* pp. 221-240; and Paul G. Hoel, *Introduction to Mathematical Statistics,* Third Edition (New York: John Wiley and Sons, Inc., 1962), pp. 335-345. For some of the tests usually encountered in econometric analysis of time series data (e.g., the Durbin-Watson test, etc.) refer, for example, to Arthur S. Goldberger, *Econometric Theory* (New York: John Wiley and Sons, 1964).

[21] Merle W. Tate and Richard C. Clelland, *Nonparametric and Shortcut Statistics* (Danville, Ill.: Interstate Printers and Publishers, Inc., 1959), p. 56.

D. DESCRIPTION OF THE EXPERIMENTS

As indicated earlier, *commodity technology* and *industry technology* models, with competitive imports treated exogenously, have been empirically developed for 1958 for the United States economy, using information from sources mentioned below. These models containing 79 x 79 order *technology* matrices, are used in obtaining predictions of intermediate demand and output levels for each sector in 1961.

The basic intersectoral flows information at the 79-sector level for 1958, for each of the two models, and the 79-order final demand and output vectors for 1961, expressed in constant 1958 dollars, are used in generating three additional models. These additional models are of the order 60 x 60, 45 x 45, and 17 x 17. Thus, a total of eight different models, four of the *commodity technology* type and the other four of the *industry technology* type, have been used in the experiments. The predictions of intermediate demand and output vectors to 1961 are compared with the actual (observed) vectors, similarly expressed in constant 1958 dollars, and appropriate measures are used to compute sectoral, as well as overall, predictive performance. The predictive measures obtained in this way are then used to study the following:

(a) the comparative predictive performance of the *commodity technology* and *industry technology* models, to find out if the two models are significantly different in terms of their predictive performance;

(b) the nature of the relationship between the level of sectoral aggregation inherent in the model and overall predictive performance, to find out the shape of the functional relationship, if there is one, and how this relationship differs as between the two models; and

(c) the structure of detailed sectoral prediction errors, to find out the extent to which they display systematic characteristics that may help us develop a better understanding of the properties of input-output systems.

The computer programs developed and used in these experiments are presented in Appendix E. They are prepared in such a way that other researchers can replicate the experiments reported here quite easily. The basic data used in the experiments are presented in Appendix F, which contains nine tables that should prove to be of considerable interest and help to other researchers. In addition, the intersectoral flows matrices, the *technology* matrices, the Leontief matrices, and the Leontief inverse matrices at the 60 x 60, 45 x 45, and 17 x 17 levels of sectoral aggregation for both the *commodity technology* and the *industry technology* model are available and can be sent to the reader upon request.

In the following few pages, we will be concerned with the following areas:

(1) a description of data sources and problems;

(2) a description of the sectoral composition of the models at different levels of aggregation;

(3) an explanation of the measures of sectoral, as well as overall, predictive performance used in the experiments; and

(4) a description of the numerical results obtained in the experiments on the predictive performance of the models and the strategy used in analyzing these results.

1. Data Sources and Problems:

a. The *Commodity Technology* Model:

The data underlying the *commodity technology* model have been made available by the National Planning Association. Both the intersectoral flows information and the *commodity technology* matrix have been presented by the National Planning Association in an unpublished report made to the U.S. Army Engineer Mathematical Computation Agency, Corps of Engineers.[22] This report also contains a detailed description of the empirical process actually followed in constructing these two matrices. The procedure described in this report represents a further development, adaptation, and implementation of a method first suggested by Edmonston.[23] In a straightforward application of Edmonston's method the inputs associated with an industry's secondary products are automatically subtracted from its input vector and then added to the input vectors of the industries producing these secondary products as their own primary product. As pointed out in this report, the mathematical application of the secondary product adjustment procedure of SPAR without

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[22] Lou P. Koenig and Philip M. Ritz, *Secondary Product Adjustment with Redistribution (SPAR)*, NREC Technical Report No. 67 (April, 1967), a Report to the U.S. Army Engineer Mathematical Computation Agency, Corps of Engineers Pursuant to Contract No. DA-18-020-ENG-3628, Task I-A, 117 pp.

[23] J. Harvey Edmonston, "A Treatment of Multiple-Process Industries," *The Quarterly Journal of Economics*, LXVI, 4 (November, 1952), 557-571.

preliminary adjustment and redefinition of some of the sectors led to very large negative entries in the flows matrix that could not be defended. In such cases, a straightforward application of the SPAR method apparently ignored the fact that much of the input structure of the secondary products of a given industry was more like the input structure of the primary products of that industry rather than like the input structure of other industries to which these secondary products are primary. As a result, a completely new set of inputs were estimated for some industries and a number of other adjustments were made which finally led to the generation of the product-to-flows matrix and the *commodity technology* matrix.[24] Under the SPAR approach, scrap products were completely omitted from the measurement of the output of each sector, as well as from the intersectoral flows matrix, and were conceptually included in each sector's value added. Further, in the SPAR listings and computations, no explicit recognition has been given to any by-products. If any by-products are produced by a sector, they are included in that sector's primary product output and are, presumably, distributed to the consumers as part of that sector's output.

Before the experiments were set up, Sectors 80 (Gross Imports of Goods and Services) and 81 (Business, Travel, Entertainment and Gifts), and 82 (Office Supplies) were omitted from analysis, since the inputs into these sectors consisted solely of secondary products transfers (i.e., fictitious rather than real economic flows).

b. The *Industry Technology* Model

The data used in developing the *industry technology* model at the 79-sectoral level have been made available by U.S. Office of Business Economics on magnetic tape (Serial No. 1239). This tape contained information on intersectoral transactions in producer's, as well as purchaser's values, distributive margins associated with each transaction, each industry's primary inputs, secondary products transfers and sales of each industry's by-products to other industries. It also contained information on the generation and consumption of scrap products and a complete listing of information on the final demand sectors. The data contained on this tape were processed, by developing and using a computer program, to generate the desired matrices and control totals. Specifically, these matrices refer to the

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[24] Koenig and Ritz, *op. cit.,* p. 18.

commodity-to-industry flows matrix, the *industry technology* matrix, the *make* matrix, and the *industry product mix* matrix, all presented in Appendix F. In addition, the data generated from this tape were used in developing a series of *dimensional measures* (e.g., primary product specialization ratio vector, etc.) that have been used in studying the structure of input-output prediction errors, as reported in Part F of this chapter.

A detailed explanation of the procedures used in generating the *commodity-to-industry flows* matrix has been documented in a working paper and can be made available to the reader upon request. In summary, by-products of a sector were shown as part of that sector's primary output and distributed to all consumers as originating from that sector. Scrap products were completely excluded from intersectoral transactions. Similarly, measures of total domestic primary product output and intermediate demand were kept free of scrap products.

c. Final Demand and Total Domestic Primary Product Output Vectors for 1961

The final demand and total domestic primary product output vectors for 1961 have been made available by U.S. Office of Business Economics in the form of a series of tabulations. The data given in these tabulations were expressed both in constant 1958 dollars, as well as in current 1961 dollars. In the experiments, only the vectors given in constant 1958 dollars have been used. The same information as contained in these tables, plus estimates of intersectoral flows in both constant 1958 and in current 1961 dollars, later became available in a report by the same agency.[25] There were some minor differences between the information given in the earlier tabulations and that contained in this report. As a result, one or two corrections were made on the earlier tabulations before using the information in the experiments. This information is presented in Appendix F, Table F-9.

It is necessary to make two comments on the 1961 information. First, the *exogenous* final demand vector used in the experiments refer to final demand minus competitive imports required for intermediate consumption. Secondly, each element in the total domestic primary product vector represents the total domestic production of a homogeneous group

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[25] U.S. Office of Business Economics, *Input-Output Transactions: 1961,* Staff Working Paper in Economics and Statistics, No. 16, July, 1968.

of products (primary products), regardless of the particular industry producing such products. This is called the *pure product* definition of sectoral output. These definitional issues have been covered fully in Chapter II, to which occasional references may be necessary.

2. Description of the Sectoral Composition of the Models at Four Different Levels of Aggregation.

The sectoral composition of the *commodity technology* and *industry technology* models developed at four different levels of aggregation is described in Tables IV-1 through IV-4. It will be noticed in Table IV-1 that the table lists 86 sectors, whereas there are only 79 sectors in the most detailed models used in the experiments. The explanation for this is rather simple. The sectors 80 through 86 represent either *dummy* sectors that are used for accounting purposes only or they contain, as in the case of noncompetitive imports, inputs that are treated outside the intersectoral flows matrix.

Aggregations to the 60 x 60, 45 x 45, and 17 x 17 sectoral levels are accomplished directly from the information base organized at the 79 x 79 level of detail. This procedure was preferred to the alternative of using the aggregated information at the 60 x 60 level in formulating the models at the 45 x 45 level, and so on, in order to minimize rounding errors that might have been introduced in such a procedure, no matter how negligible they may have been.

The sectors have been aggregated on the basis of the criterion that the individual sectors that are combined at each level of aggregation display homogeneity in terms of their products and input structure (i.e., production process or technology). The decisions made on which particular sectors should be combined to go down the scale to a new level of sectoral aggregation were based on the author's personal experience and knowledge in the input-output field during the past six years, and do not represent aggregations arrived at through the application of optimal aggregation procedures. Application of optimal aggregation procedures would have missed the point of part of the purpose of conducting these experiments. Here, the question of interest is not how the alternative *optimal* aggregation procedures compare with one another, but rather what the empirical nature of the *aggregation bias* is, when the aggregation process is accomplished by applying a set of rules based mostly on experience and specialized knowledge, and further, how the *aggregation bias* varies as between the *commodity technology* and *industry technology* models when the same set of aggregation procedures are

TABLE IV-1

LIST OF THE ECONOMIC SECTORS USED IN THE EXPERIMENTS

Industry No. and industry title	Related SIC codes (1957 edition)
Agriculture, forestry and fisheries	
1 Livestock and livestock products	013, pt. 014, 0193, pt. 02, pt. 0729.
2 Other agricultural products	011, 012, pt. 014, 0192, 0199, pt. 02.
3 Forestry and fishery products	074, 081, 082, 084, 086, 091.
4 Agricultural, forestry and fisheries services.	071, 0723, pt. 0729, 085, 098.
Mining	
5 Iron and ferroalloy ores mining	1011, 106.
6 Nonferrous metal ores mining	102, 103, 104, 105, 108, 109.
7 Coal mining	11, 12.
8 Crude petroleum and natural gas	1311, 1321.
9 Stone and clay mining and quarrying	141, 142, 144, 145, 148, 149.
10 Chemical and fertilizer mineral mining.	147.
Construction	
11 New construction	138, pt. 15, pt. 16, pt. 17, pt. 6561.
12 Maintenance and repair construction	pt. 15, pt. 16, pt. 17.
Manufacturing	
13 Ordnance and accessories	19.
14 Food and kindred products	20.
15 Tobacco manufactures	21.
16 Broad and narrow fabrics, yarn and thread mills.	221, 222, 223, 224, 226, 228.
17 Miscellaneous textile goods and floor coverings.	227, 229.
18 Apparel	225, 23 (excluding 239), 3992.
19 Miscellaneous fabricated textile products.	239.
20 Lumber and wood products, except containers.	24 (excluding 244).
21 Wooden containers	244.
22 Household furniture	251.
23 Other furniture and fixtures	25 (excluding 251).
24 Paper and allied products, except containers and boxes.	26 (excluding 265).
25 Paperboard containers and boxes	265.
26 Printing and publishing	27.
27 Chemicals and selected chemical products.	281 (excluding alumina pt. of 2819), 286, 287, 289.
28 Plastics and synthetic materials	282.
29 Drugs, cleaning, and toilet preparations.	283, 284.
30 Paints and allied products	285.
31 Petroleum refining and related industries.	29.
32 Rubber and miscellaneous plastics products.	30.
33 Leather tanning and industrial leather products.	311, 312.
34 Footwear and other leather products	31 (excluding 311, 312).
35 Glass and glass products	321, 322, 323.
36 Stone and clay products	324, 325, 326, 327, 328, 329.
37 Primary iron and steel manufacturing.	331, 332, 3391, 3399.
38 Primary nonferrous metals manufacturing.	2819 (alumina only), 333, 334, 335, 336, 3392.
39 Metal containers	3411, 3491.
40 Heating, plumbing and fabricated structural metal products.	343, 344.
41 Screw machine products, bolts, nuts, etc., and metal stampings.	345, 346.
42 Other fabricated metal products	342, 347, 348, 349 (excluding 3491).
43 Engines and turbines	351.
44 Farm machinery and equipment	352.
45 Construction, mining, oil field machinery and equipment.	3531, 3532, 3533.
46 Materials handling machinery and equipment.	3534, 3535, 3536, 3537.
47 Metalworking machinery and equipment.	354.
48 Special industry machinery and equipment.	355.
49 General industrial machinery and equipment.	356.
50 Machine shop products	359.
51 Office, computing and accounting machines.	357.
52 Service industry machines	358.
53 Electric transmission and distribution equipment, and electrical industrial apparatus.	361, 362.
54 Household appliances	363.
55 Electric lighting and wiring equipment.	364.
56 Radio, television, and communication equipment.	365, 366.
57 Electronic components and accessories.	367.
58 Miscellaneous electrical machinery, equipment and supplies.	369.
59 Motor vehicles and equipment	371.
60 Aircraft and parts	372.
61 Other transportation equipment	373, 374, 375, 379.
62 Professional, scientific, and controlling instruments and supplies.	381, 382, 384, 387.
63 Optical, ophthalmic, and photographic equipment and supplies.	383, 385, 386.
64 Miscellaneous manufacturing	39 (excluding 3992).
Transportation, communication, electric, gas, and sanitary services	
65 Transportation and warehousing	40, 41, 42, 44, 45, 46, 47.
66 Communications, except radio and television broadcasting.	481, 482, 489.
67 Radio and T.V. broadcasting	483.
68 Electric, gas, water, and sanitary services.	49.
Wholesale and retail trade	
69 Wholesale and retail trade	50 (excluding manufacturers sales offices), 52, 53, 54, 55, 56 57, 58, 59, pt. 7399.
Finance insurance and real estate	
70 Finance and insurance	60, 61, 62, 63, 64, 66, 67.
71 Real estate and rental	65 (excluding 6541 and pt. 6561).
Services	
72 Hotels and lodging places; personal and repair services, except automobile repair.	70, 72, 76 (excluding 7694 and 7699).
73 Business services	6541, 73 (excluding 7361, 7391, and pt. 7399), 7694, 7699, 81, 89 (excluding 8921).
74 Research and development	----------------
75 Automobile repair and services	75.
76 Amusements	78, 79.
77 Medical, educational services, and nonprofit organizations.	0722, 7361, 80, 82, 84, 86, 8921.
Government enterprises	
78 Federal Government enterprises	----------------
79 State and local government enterprises.	----------------
Imports	
80 Gross imports of goods and services	----------------
Dummy industries	
81 Business travel, entertainment, and gifts.	----------------
82 Office supplies	----------------
83 Scrap, used and secondhand goods	----------------
Special industries	
84 Government industry	----------------
85 Rest of the world industry	----------------
86 Household industry	----------------

Source: National Economics Division Staff, "The Transactions Table of the 1958 Input-Output Study and Revised Direct and Total Requirements Data," *Survey of Current Business,* XLV, 9 (September, 1965), 33.

TABLE IV-2

AGGREGATION OF SECTORS FROM THE 79 x 79 SYSTEM INTO THE 60 x 60 SYSTEM
(The same procedure applies to both matrices and vectors)

Sector No.	Sector Description	Corresponding Sectors of the 79 x 79 System That Are Combined
1	Livestock and Other Agricultural Products	1,2
2	Forestry and Fishery Products and Services	3,4
3	Mining, Metal	5,6
4	Coal Mining	7
5	Petroleum and Related Industries	8,31
6	Stone and Clay Mining and Quarrying	9
7	Construction	11, 12
8	Ordnance and Accessories	13
9	Food and Kindred Products	14
10	Tobacco Manufactures	15
11	Fabrics, Apparel, Textiles	16, 17, 18, 19
12	Lumber and Wood Products	20, 21
13	Furniture and Fixtures	22, 23
14	Printing and Publishing	26
15	Paper and Allied Products, Containers	24, 25
16	Chemicals	10, 27
17	Plastics and Synthetic Materials	28
18	Drugs, Cleaning and Toilet Preparations	29
19	Paints and Allied Products	30
20	Rubber and Miscellaneous Plastics Products	32
21	Leather Tanning, Footwear, and Other Leather Products	33, 34
22	Glass, Stone, and Clay Products	35, 36
23	Primary Iron and Steel Manufacturing	37
24	Primary Nonferrous Metals Manufacturing	38
25	Metal Containers	39
26	Heating, Plumbing, and Structural Metal Products	40
27	Screw Machine Products and Metal Stampings	41
28	Other Fabricated Metal Products	42
29	Engines and Turbines	43
30	Farm Machinery and Equipment	44
31	Construction, Mining, and Oil Field Machinery and Equipment	45
32	Materials Handling Machinery and Equipment	46
33	Metalworking Machinery and Equipment	47
34	Special Industry Machinery and Equipment	48
35	General Industry Machinery and Equipment	49
36	Machine Shop Products	50
37	Office, Computing, and Accounting Machines	51
38	Service Industry Machines	52
39	Electrical Equipment and Apparatus, Electric Lighting and Wiring Equipment	53, 55
40	Household Appliances	54
41	Radio, TV, and Communication Equipment	56
42	Electronic Components and Accessories	57
43	Miscellaneous Electrical Machinery, Equipment and Supplies	58
44	Motor Vehicles and Equipment	59
45	Aircraft and Parts	60
46	Other Transportation Equipment	61
47	Instruments, Optical Goods, and Photographic Equipment	62, 63
48	Miscellaneous Manufacturing	64
49	Transportation and Warehousing	65
50	Communications, Radio and TV Broadcasting	66,67
51	Electric, Gas, Water, and Sanitary Services	68
52	Wholesale and Retail Trade	69
53	Finance and Insurance	70
54	Real Estate and Rental	71
55	Hotels, Personal and Repair Services, Automobile Repair and Services	72, 75
56	Business Services	73
57	Research and Development	74
58	Amusements	76
59	Medical, Educational Services, Nonprofit Organizations	77
60	Federal, State, and Local Government Enterprises	78,79

TABLE IV-3

AGGREGATION OF SECTORS FROM THE 79 x 79 SYSTEM INTO THE 45 x 45 SYSTEM
(the same procedure applies to both matrices and vectors)

Sector No.	Sector Description	Corresponding Sectors of the 79 x 79 System That Are Combined
1	Agriculture, Forestry, Fisheries	1, 2, 3, 4
2	Mining (Metals)	5, 6
3	Coal Mining	7
4	Petroleum and Related Industries	8, 31
5	Stone and Clay Mining and Quarrying	9
6	Construction	11, 12
7	Ordnance and Accessories	13
8	Food and Kindred Products	14
9	Tobacco Manufactures	15
10	Fabrics, Apparel, Textiles	16, 17, 18, 19
11	Lumber and Wood Products	20, 21
12	Furniture and Fixtures	22, 23
13	Printing and Publishing	26
14	Paper and Allied Products, Containers	24, 25
15	Chemicals, Plastics, Drugs, Paints and Allied Products	10, 27, 28, 29, 30
16	Rubber and Miscellaneous Plastics Products	32
17	Leather, Footwear, and Related Products	33, 34
18	Glass, Stone, and Clay Products	35, 36
19	Primary Iron and Steel Manufacturing	37
20	Primary Nonferrous Metals Manufacturing	38
21	Fabricated Metals (Including Metal Containers)	39, 40, 41, 42
22	Engines and Turbines	43
23	Farm Machinery and Equipment	44
24	Construction, Mining, and Oil Field Machinery and Equipment	45
25	Materials Handling and Metalworking Machinery and Equipment	46, 47
26	Special and General Industrial Machinery and Equipment, Machine Shop Products	48, 49, 50
27	Office, Computing, and Accounting Machines, Service Industry Machines	51, 52
28	Electrical Equipment and Apparatus, Electric Lighting and Wiring Equipment	53, 55
29	Household Appliances	54
30	Radio, TV, Communication Equipment, Electronic Components, and Miscellaneous Electrical Products	56, 57, 58
31	Motor Vehicles and Equipment	59
32	Aircraft and Parts and Other Transportation Equipment	60, 61
33	Instruments, Optical, Photographic Products, and Miscellaneous Manufacturing	62, 63, 64
34	Transportation and Warehousing	65
35	Communications, Radio and TV Broadcasting	66, 67
36	Electric, Gas, Water and Sanitary Services	68
37	Wholesale and Retail Trade	69
38	Finance and Insurance	70
39	Real Estate and Rental	71
40	Hotels, Personal and Repair Services, Automobile Repair and Services	72, 75
41	Business Services	73
42	Research and Development	74
43	Amusements	76
44	Medical, Educational Services, Nonprofit Organizations	77
45	Federal, State, and Local Government Enterprises	78, 79

TABLE IV-4

AGGREGATION OF SECTORS FROM THE 79 x 79 SYSTEM INTO THE 17 x 17 SYSTEM

(the same procedure applies to both matrices and vectors)

Sector No.	Sector Description	Corresponding Sectors of the 79 x 79 System That Are Combined
1	Agriculture, Food, Tobacco	1, 2, 3, 4, 14, 15
2	Construction and Related Industries	9, 11, 12, 35, 36
3	Energy, Including Utilities	7, 8, 31, 68
4	Paper, Printing and Publishing	24, 25, 26
5	Furniture, Lumber and Wood Products	20, 21, 22, 23
6	Chemicals, Textiles, Rubber, Synthetics, Related Products	10, 16, 17, 18, 19, 27, 28, 29, 30, 32, 33, 34
7	Metals	5, 6, 37, 38, 39, 40, 41, 42
8	Machinery, Instruments, and Miscellaneous Manufacturing	13, 43-49, 50-58, 62, 63, 64
9	Transportation Equipment	59, 60, 61
10	Transportation and Warehousing	65
11	Wholesale and Retail Trade	69
12	Communications and Broadcasting	66, 67
13	Finance and Insurance	70
14	Real Estate and Rental	71
15	Business Services, Research and Development	73, 74
16	Personal Services	72, 75, 76, 77
17	Government Enterprises	78, 79

applied to them both. Thus, the aggregation process used here would seem to represent a reasonable sample from a population of possible non-optimal aggregation procedures. Another researcher may wish to apply his own aggregation rules to see how his results compare with those obtained in the experiments reported in this chapter.

3. Measures of Detailed and Overall Predictive Performance Used in the Experiments

A list of the detailed, as well as overall, measures of predictive performance developed and used in the experiments can be given as follows:

a. Detailed Measures of Predictive Performance (Computed for Each Sector):

(1) Relative Prediction Error (Computed for Each Sector):

$$e_i = \frac{|\hat{z}_i - z_i|}{z_i} \qquad i = 1,2,\ldots,n; \tag{4.48}$$

where

$\hat{z}_i$ is the predicted intermediate demand or domestic primary product output level for sector i;

z_i is the actual (observed) 1961 intermediate demand or domestic primary product output level, expressed in constant 1958 dollars.

(2) Logarithmic Prediction Error (Computed for Each Sector):

$$\theta_i = \log \frac{\hat{z}_i}{z_i} \qquad i = 1,2,\ldots,n; \tag{4.49}$$

where natural logarithms are used.

b. Overall Measures of Predictive Performance:

(1) Mean Relative Prediction Error

$$e = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{|\hat{z}_i - z_i|}{z_i} \right) \tag{4.50}$$

(2) Root Mean Square Prediction Error

$$\text{RMS} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\hat{z}_i - z_i)^2} \tag{4.51}$$

(3) Weighted Root Mean Square Prediction Error:

(4.52)
$$RMS^* = \sqrt{\frac{\sum_{i=1}^{n} (\hat{z}_i - z_i)^2 \, z_i}{\sum_{i=1}^{n} z_i}}$$

(4) Mean Logarithmic Prediction Error

(4.53)
$$\theta = \frac{1}{n} \sum_{i=1}^{n} \log \frac{\hat{z}_i}{z_i}$$

(5) Weighted Mean Logarithmic Prediction Error

(4.54)
$$\theta^* = \frac{1}{\sum_{i=1}^{n} z_i} \left[\sum_{i=1}^{n} \log \frac{z_i}{z_i} \cdot z_i \right]$$

(6) Pearsonian (Product-moment) Coefficient of Correlation

(4.55)
$$r = \frac{n \cdot \sum_{i=1}^{n} z_i \hat{z}_i - \sum_{i=1}^{n} z_i \sum_{i=1}^{n} \hat{z}_i}{[n \cdot \sum_{i=1}^{n} z_i^2 - (\sum_{i=1}^{n} z_i)^2]^{1/2} \, [n \cdot \sum_{i=1}^{n} \hat{z}_i^2 - (\sum_{i=1}^{n} \hat{z}_i)^2]^{1/2}}$$

The first five measures are measures of location or mean predictive performance, while the sixth is a measure of the *goodness of fit* between predictions and realizations (i.e., actual observations, referring to 1961). Other measures of overall predictive performance have been used by various authors. Among these, we can mention Theil's Coefficient of Inequality[26] and Wold's Janus Coefficient.[27]

Among the measures used here, the RMS prediction error measure has been used by Theil and his associates in analyzing the quality of input-output predictions.[28] Also, in his input-output experiments, Tilanus used (natural) logarithmic prediction errors, as formulated here, in studying the structure of input-output prediction errors.[29] The rationale given by Tilanus in using logarithmic prediction errors can be briefly repeated here for the purpose of clarification.[30]

The relative prediction error, as defined above, is a common measure of the quality of predictions, but it has the disadvantage that, apart from the sign, it is not symmetric in prediction and realization. Suppose, for example, that the prediction for the output of a given sector is 7.03 and that the realization is 8.0, which results in a relative error of minus 12 percent. If, conversely, the prediction had been 8.0 and the realization 7.03, then the relative prediction error would have been not minus 12 percent but plus 14 percent. The symmetry

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[26] H. Theil, *Economic Forecasts and Policy,* Second Edition (Amsterdam: North-Holland Publishing Co., 1961), p. 32. Theil later gave a new definition of the Inequality Coefficient in his *Applied Economic Forecasting* (Amsterdam: North-Holland Publishing Co., and Chicago: Rand McNally and Co., 1966), p. 28. In its new version, the Inequality Coefficient is expressed as follows:

$$U^2 = \frac{\sum_{i=1}^{n} (P_i - A_i)^2}{\sum_{i=1}^{n} A_i^2}$$

where (P_i, A_i) stands for a pair of predicted and observed *changes*. More correctly, the term Inequality Coefficient is applied to the term U, which is zero only when the forecasts are all perfect, and is equal to unity when the prediction procedure leads to the same RMS error as naive no-change extrapolation.

[27] Herman, O.A. Wold (ed.), *Econometric Model Building* (Amsterdam: North-Holland Publishing Co., 1964), pp. 229-235. The discussion of the Janus Coefficient given here includes comparisons with Theil's original Coefficient of Inequality.

[28] Theil, *op. cit. [Applied Economic Forecasting]*, pp. 178-190.

[29] C.B. Tilanus, *Input-Output Experiments, The Netherlands, 1948-1961* (Rotterdam: Rotterdam University Press, 1966).

[30] See *Ibid,* pp. 13-17.

between prediction and realization can be restored by using logarithmic prediction errors, defined as the natural logarithm of the ratio between prediction and realization. Logarithmic errors are algebraically smaller than relative errors. This can be illustrated as follows. Let the relative error

$$\frac{\text{prediction} - \text{realization}}{\text{realization}} \tag{4.56}$$

where we do not take the absolute value of the difference in the numerator but simply the algebraic difference, be denoted by r. Then, the logarithmic error can be written as

$$e = \log (1 + r). \tag{4.57}$$

The relative error minus the logarithmic error

$$r - \log (1 + r) \tag{4.58}$$

has a first derivative

$$1 - \frac{1}{1 + r} \tag{4.59}$$

which is zero for r and e equal to zero. Its second derivative

$$\frac{1}{(1 + r)^2} \tag{4.60}$$

is positive. Therefore, the function $r - \log (1 + r)$ has a minimum value of zero at point zero and is positive elsewhere. Hence, $\log (1 + r) < r$, except for the trivial case $r = 0$.

4. Description of Numerical Results on the Predictive Performance of the Models and the Approach Used in Analyzing Them.

The numerical results of the experiments are summarized in Tables IV-5 through IV-10, where the first two tables contain observed measures of overall predictive performance for both the *commodity technology* and the *industry technology* model at four different levels of sectoral aggregation, and the last four tables contain a listing of logarithmic prediction errors for each individual sector for both models at each aggregation level. The relative prediction errors have also been computed, but are not presented here at the level

TABLE IV-5

COMPARATIVE PREDICTIVE PERFORMANCE OF COMMODITY TECHNOLOGY AND INDUSTRY TECHNOLOGY MODELS AT DIFFERENT LEVELS OF SECTORAL AGGREGATION, WITH COMPETITIVE IMPORTS TREATED EXOGENOUSLY

(Prediction of intermediate demand levels to 1961)

		Aggregation I (79x79)		Aggregation II (60x60)		Aggregation III (45x45)		Aggregation IV (17x17)	
		"Commodity Technology" Model	"Industry Technology" Model	"Commodity Technology" Model	"Industry Technology" Model	"Commodity Technology" Model	"Industry Technology" Model	"Commodity Technology" Model	"Industry Technology" Model
e:	Mean Relative Prediction Error	0.12607	0.06790	0.12307	0.08604	0.14148	0.09254	0.15304	0.07162
θ:	Mean Logarithmic Prediction Error	0.00333	0.02246	0.01656	0.04760	0.02486	0.02765	0.07393	0.01964
θ^*:	Weighted Mean Logarithmic Prediction Error	0.02057	0.03018	0.02068	0.04365	0.09504	0.06153	0.24129	0.10094
RMS:	Root Mean Square Prediction Error	1453.1	419.9	1776.9	630.8	3490.4	2060.6	15194.8	8433.1
RMS^*:	Weighted Root Mean Square Prediction Error	2065.6	754.2	2439.9	1048.4	6111.7	3937.2	28359.0	3918.5
r :	Pearsonian (Product-moment) Coefficient of Correlation	.97485	.99855	.97552	.99810	.93624	.98022	.77007	.95662

TABLE IV-6

COMPARATIVE PREDICTIVE PERFORMANCE OF COMMODITY TECHNOLOGY AND INDUSTRY TECHNOLOGY MODELS AT DIFFERENT LEVELS OF SECTORAL AGGREGATION, WITH COMPETITIVE IMPORTS TREATED EXOGENOUSLY

(Prediction of sectoral product output levels)

	Aggregate Measures of Predictive Performance	Aggregation I (79x79)		Aggregation II (60x60)		Aggregation III (45x45)		Aggregation IV (17x17)	
		"Commodity Technology" Model	"Industry Technology" Model	"Commodity Technology" Model	"Industry Technology" Model	"Commodity Technology" Model	"Industry Technology" Model	"Commodity Technology" Model	"Industry Technology" Model
e:	Mean Relative Prediction Error	0.07367	0.03283	0.06626	0.04241	0.07431	0.04218	0.10076	0.04504
θ:	Mean Logarithmic Prediction Error	0.00545	0.01538	0.00289	0.02594	0.00442	0.01412	0.03096	0.01412
θ^*:	Weighted Mean Logarithmic Prediction Error	0.00747	0.01345	0.00743	0.01930	0.03633	0.02590	0.09713	0.04433
RMS:	Root Mean Square Prediction Error	1453.1	419.9	1776.9	630.8	3490.4	2060.6	15194.8	8433.1
RMS^*:	Weighted Root Mean Square Prediction Error	1625.8	776.4	1884.7	987.8	5319.6	3493.5	22926.6	13067.6
r :	Pearsonian (Product-moment) Coefficient of Correlation	.99640	.99978	.99614	.99962	.98872	.99622	1.00000	.97918

TABLE IV-7

COMPARATIVE PREDICTIVE PERFORMANCE OF THE "COMMODITY TECHNOLOGY" AND "INDUSTRY TECHNOLOGY" MODELS BY SECTOR, 1958-1961, AT THE 79X79 LEVEL OF SECTORAL AGGREGATION

(Prediction error measured as $\theta_i = \log \dfrac{\hat{z}_i}{z_i}$)[c]

Sector No.	"Commodity Technology" Model: Output Prediction[a]	"Commodity Technology" Model: Intermediate Demand Prediction[b]	"Industry Technology" Model: Output Prediction	"Industry Technology" Model: Intermediate Demand Prediction
1	0.00989	0.01108	0.00172	0.00192
2	0.01972	0.02572	0.01700	0.02215
3	0.03858	0.04643	0.00558	0.00674
4	0.01655	0.01622	0.00912	0.00894
5	0.01638	0.01758	0.00142	0.00152
6	0.05482	0.06912	0.07405	0.09314
7	0.08641	0.10337	0.06855	0.08215
8	0.04498	0.04536	0.05171	0.05215
9	0.05166	0.05269	0.06239	0.06364
10	0.01508	0.02138	0.03265	0.04677
11	0.0	0.0	0.0	0.0
12	0.03723	0.05090	0.03301	0.04518
13	0.03167	0.29302	0.00182	0.01472
14	0.00339	0.01546	0.00580	0.02605
15	0.01505	0.08899	0.01494	0.08838
16	0.01379	0.01538	0.00459	0.00512
17	0.09288	0.15381	0.10062	0.16706
18	0.00765	0.03981	0.00727	0.03783
19	0.01422	0.03265	0.01528	0.03510
20	0.01880	0.01931	0.03109	0.03195
21	0.03183	0.03208	0.03246	0.03272
22	0.02891	0.16616	0.03202	0.18273
23	0.02432	0.13359	0.02649	0.14483
24	0.29927	0.34542	0.04889	0.05808
25	0.89501	0.90717	0.07146	0.07299
26	0.74142	1.12450	0.04273	0.08243
27	0.04872	0.05952	0.07782	0.09537
28	0.09199	0.10599	0.10935	0.12616
29	0.01827	0.06758	0.02095	0.07781
30	0.01468	0.01518	0.01522	0.01574
31	0.00032	0.00069	0.00760	0.01674
32	0.12378	0.17201	0.12962	0.18034
33	0.03931	0.04142	0.03535	0.03732
34	0.00227	0.02274	0.00289	0.02884
35	0.06835	0.07474	0.06094	0.06661
36	0.02984	0.03142	0.02707	0.02850
37	0.02659	0.02783	0.01894	0.01980
38	0.08241	0.08944	0.04869	0.05277
39	0.03761	0.03891	0.03500	0.03620
40	0.00959	0.01087	0.00945	0.01071
41	0.04930	0.05515	0.07415	0.08283
42	0.01055	0.01257	0.01025	0.01222
43	0.04936	0.11723	0.05225	0.12386
44	0.00457	0.02556	0.01121	0.06366
45	0.04655	0.18997	0.04519	0.18477
46	0.06238	0.15524	0.06301	0.15690
47	0.02134	0.04335	0.04602	0.09232
48	0.01072	0.06581	0.00939	0.06080
49	0.07328	0.12615	0.05955	0.10200
50	0.27931	0.29542	0.13418	0.14133
51	0.01871	0.12994	0.02721	0.19417
52	0.03834	0.13605	0.04229	0.15087
53	0.05584	0.10419	0.04086	0.07574
54	0.01798	0.12619	0.01730	0.12114
55	0.03791	0.04865	0.02429	0.03111
56	0.04122	0.27131	0.01843	0.11369
57	0.08153	0.10179	0.05046	0.06275
58	0.06607	0.10828	0.05660	0.09247
59	0.01608	0.04433	0.01696	0.04682
60	0.01947	0.07826	0.05248	0.19086
61	0.01976	0.10394	0.02144	0.11318
62	0.03952	0.10326	0.01557	0.03991
63	0.01228	0.03680	0.00027	0.00082
64	0.04932	0.17061	0.04810	0.16614
65	0.03287	0.05615	0.04668	0.08014
66	0.01859	0.04154	0.01841	0.04114
67	0.04307	0.12149	0.02945	0.08408
68	0.05123	0.09185	0.05714	0.10269
69	0.01019	0.03697	0.01359	0.04953
70	0.02870	0.05544	0.02435	0.04713
71	0.01433	0.04454	0.01643	0.05119
72	0.01140	0.12757	0.01054	0.11836
73	0.27979	0.33182	0.03313	0.03843
74	0.00156	0.05632	0.00110	0.03935
75	0.02401	0.05735	0.02883	0.06910
76	0.06786	0.21833	0.00911	0.02699
77	0.00127	0.02314	0.00121	0.02207
78	0.03028	0.04044	0.00111	0.00149
79	0.02484	0.06651	0.01637	0.04413

NOTES: [a]Sectoral output is defined here in pure product terms (i.e., the "primary" products of a given sector, plus the "secondary" products of all other industries that are primary to the sector in question). For further definitional details, refer to Chapter II.

[b]Intermediate demand refers to the interindustry shipments of a homogeneous product group, domestically produced, regardless of which industry has actually produced it.

[c]Prediction error is measured by θ_i, where

θ_i: logarithmic prediction error for sector i, using natural logarithms.

z_i: actual domestic output of (or intermediate demand for) product class i (i = 1,...,n) in 1961, expressed in constant 1958 dollars.

$\hat{z}_i$: predicted domestic output of (or intermediate demand for) product class i (i = 1,...,n) in 1961, expressed in constant 1958 dollars.

TABLE IV-8

COMPARATIVE PREDICTIVE PERFORMANCE OF THE "COMMODITY TECHNOLOGY" AND "INDUSTRY TECHNOLOGY" MODELS BY SECTOR, 1958-1961, AT THE 60X60 LEVEL OF SECTORAL AGGREGATION

(Prediction error measured as $\theta_i = \log \frac{\hat{z}_i}{z_i}$)[c]

Sector No.	"Commodity Technology" Model: Output Prediction[a]	"Commodity Technology" Model: Intermediate Demand Prediction[b]	"Industry Technology" Model: Output Prediction	"Industry Technology" Model: Intermediate Demand Prediction
1	0.01886	0.02259	0.03913	0.04696
2	0.01899	0.02022	0.04141	0.04418
3	0.01718	0.02018	0.04584	0.05372
4	0.09408	0.11247	0.10877	0.12984
5	0.01004	0.01534	0.00240	0.00366
6	0.04095	0.04176	0.05327	0.05433
7	0.00874	0.04825	0.01181	0.06472
8	0.03165	0.29271	0.02938	0.26896
9	0.00420	0.01917	0.00392	0.01765
10	0.01505	0.08899	0.01487	0.08797
11	0.00259	0.00524	0.00530	0.01079
12	0.05434	0.05572	0.03809	0.03911
13	0.02831	0.15998	0.02682	0.15205
14	0.45733	0.50441	0.17539	0.20089
15	0.74090	1.12384	0.18378	0.38208
16	0.07816	0.13290	0.03423	0.05731
17	0.08677	0.09993	0.11107	0.12817
18	0.01805	0.06676	0.01291	0.04740
19	0.02948	0.03049	0.03119	0.03227
20	0.12176	0.16913	0.12062	0.16751
21	0.01206	0.04148	0.04640	0.17222
22	0.03615	0.03840	0.03667	0.03896
23	0.02641	0.02764	0.01088	0.01138
24	0.08346	0.09058	0.08730	0.09476
25	0.04054	0.04194	0.01678	0.01735
26	0.01275	0.01446	0.01563	0.01772
27	0.04921	0.05505	0.03382	0.03786
28	0.01579	0.01880	0.00190	0.00227
29	0.04956	0.11770	0.07684	0.17928
30	0.00270	0.01504	0.01791	0.09531
31	0.04668	0.19045	0.05342	0.21585
32	0.06009	0.14929	0.05551	0.13742
33	0.02125	0.04316	0.01875	0.03814
34	0.00498	0.03102	0.00242	0.01518
35	0.07329	0.12617	0.07598	0.13094
36	0.28118	0.29741	0.26274	0.27775
37	0.01880	0.13061	0.02855	0.20459
38	0.03736	0.13241	0.01873	0.06482
39	0.04952	0.08109	0.05472	0.08976
40	0.01757	0.12315	0.01806	0.12677
41	0.03919	0.25639	0.04265	0.28195
42	0.08100	0.10112	0.11932	0.14969
43	0.06743	0.11055	0.07603	0.12501
44	0.01710	0.04720	0.02461	0.06838
45	0.01948	0.07827	0.01611	0.06441
46	0.01998	0.10515	0.00149	0.00755
47	0.02331	0.06323	0.01891	0.05109
48	0.04542	0.15635	0.03882	0.13252
49	0.03314	0.05660	0.05604	0.09654
50	0.01371	0.03056	0.01847	0.04128
51	0.05011	0.08981	0.02996	0.05325
52	0.01039	0.03769	0.01543	0.05637
53	0.02894	0.05591	0.02676	0.05175
54	0.01354	0.04205	0.01549	0.04821
55	0.00297	0.01321	0.00694	0.03111
56	0.27607	0.32730	0.03181	0.03689
57	0.00172	0.06208	0.00002	0.00975
58	0.00095	0.00285	0.00234	0.00697
59	0.00142	0.02583	0.00250	0.04597
60	0.02994	0.04436	0.06883	0.10107

NOTES: [a]Sectoral output is defined here in pure product terms (i.e., the "primary" products of a given sector, plus the "secondary" products of all other industries that are primary to the sector in question. For further definitional details, refer to Chapter II.

[b]Intermediate demand refers to the interindustry shipments of a homogeneous product group, domestically produced, regardless of which industry has actually produced it.

[c]Prediction error is measured by θ_i, where

θ_i: logarithmic prediction error for sector i, using natural logarithms.

z_i: actual domestic output of (or intermediate demand for) product class i (i = 1, ..., n) in 1961, expressed in constant 1958 dollars.

$\hat{z}_i$: predicted domestic output of (or intermediate demand for) product class i (i = 1, ..., n) in 1961, expressed in constant 1958 dollars.

TABLE IV-9

COMPARATIVE PREDICTIVE PERFORMANCE OF THE "COMMODITY TECHNOLOGY" AND "INDUSTRY TECHNOLOGY" MODELS BY SECTOR, 1958-1961, AT THE 45X45 LEVEL OF SECTORAL AGGREGATION

(Prediction error measured as $\theta_i = \log \frac{\hat{z}_i}{z_i}$)[c]

Sector No.	"Commodity Technology" Model: Output Prediction[a]	"Commodity Technology" Model: Intermediate Demand Prediction[b]	"Industry Technology" Model: Output Prediction	"Industry Technology" Model: Intermediate Demand Prediction
1	0.01880	0.02236	0.03442	0.04100
2	0.00782	0.00921	0.08436	0.09854
3	0.07228	0.08659	0.13853	0.16490
4	0.02080	0.03187	0.01351	0.02051
5	0.05318	0.05423	0.03340	0.03406
6	0.00764	0.04228	0.01361	0.07432
7	0.02974	0.27273	0.02712	0.24582
8	0.00635	0.02912	0.00835	0.03731
9	0.01505	0.08899	0.01487	0.08799
10	0.00106	0.00216	0.00198	0.00402
11	0.04309	0.04419	0.02732	0.02804
12	0.02481	0.14129	0.02400	0.13693
13	0.41873	0.46268	0.11800	0.13458
14	0.70274	1.07434	0.17311	0.35764
15	0.39703	0.81705	0.25346	0.48035
16	0.13114	0.18250	0.10290	0.14240
17	0.01284	0.04411	0.04483	0.16605
18	0.04589	0.04876	0.01745	0.01852
19	0.03215	0.03365	0.02416	0.02525
20	0.08623	0.09360	0.06776	0.07349
21	0.00374	0.00426	0.02599	0.02953
22	0.04322	0.10309	0.07853	0.18302
23	0.00464	0.02593	0.02291	0.12063
24	0.04244	0.17424	0.05876	0.23565
25	0.00012	0.00027	0.00858	0.01808
26	0.09386	0.18746	0.07456	0.14744
27	0.03148	0.14639	0.01992	0.09065
28	0.05395	0.08848	0.05540	0.09089
29	0.01811	0.12711	0.01819	0.12773
30	0.06405	0.17417	0.07282	0.19960
31	0.01735	0.04791	0.02395	0.06652
32	0.01752	0.07440	0.00859	0.03595
33	0.03604	0.10919	0.02424	0.07256
34	0.04521	0.07756	0.03821	0.06539
35	0.01984	0.04438	0.00956	0.02125
36	0.05964	0.10729	0.01596	0.02822
37	0.01404	0.05119	0.00974	0.03531
38	0.02344	0.04540	0.03541	0.06820
39	0.01652	0.05148	0.01046	0.03237
40	0.00489	0.02186	0.00388	0.01732
41	0.29609	0.35172	0.00632	0.00731
42	0.00378	0.14168	0.00279	0.09344
43	0.00033	0.00098	0.00328	0.00977
44	0.00198	0.03633	0.00162	0.02965
45	0.01750	0.02601	0.08968	0.13105

NOTES: [a]Sectoral output is defined here in pure product terms (i.e., the "primary" products of a given sector, plus the "secondary" products of all other industries that are primary to the sector in question. For further definitional details, refer to Chapter II.

[b]Intermediate demand refers to the interindustry shipments of a homogeneous product group, domestically produced, regardless of which industry has actually produced it.

[c]Prediction error is measured by θ_i, where

θ_i: logarithmic prediction error for sector i, using natural logarithms.

z_i: actual domestic output of (or intermediate demand for) product class i (i = 1, ..., n) in 1961, expressed in constant 1958 dollars.

$\hat{z}_i$: predicted domestic output of (or intermediate demand for) product class i (i = 1, ..., n) in 1961, expressed in constant 1958 dollars.

TABLE IV-10

COMPARATIVE PREDICTIVE PERFORMANCE OF THE "COMMODITY TECHNOLOGY" AND "INDUSTRY TECHNOLOGY" MODELS BY SECTOR, 1958-1961, AT THE 17X17 LEVEL OF SECTORAL AGGREGATION

(Prediction error measured as $\theta_i = \log \frac{\hat{z}_i}{z_i}$)[c]

Sector No.	"Commodity Technology" Model: Output Prediction[a]	"Commodity Technology" Model: Intermediate Demand Prediction[b]	"Industry Technology" Model: Output Prediction	"Industry Technology" Model: Intermediate Demand Prediction
1	0.01068	0.02302	0.01383	0.02988
2	0.00499	0.01756	0.01600	0.05494
3	0.05023	0.08148	0.04152	0.06552
4	0.54049	0.66089	0.16304	0.21883
5	0.01357	0.01975	0.00563	0.00822
6	0.04008	0.06910	0.02913	0.05001
7	0.58732	0.95764	0.29362	0.43538
8	0.04568	0.12333	0.00073	0.00190
9	0.02052	0.06452	0.01682	0.05267
10	0.06709	0.11605	0.00755	0.01278
11	0.02107	0.07756	0.00303	0.01082
12	0.02068	0.04629	0.01299	0.02850
13	0.01883	0.03654	0.05140	0.09830
14	0.01695	0.05283	0.00154	0.00470
15	0.22825	0.34403	0.00414	0.00590
16	0.00437	0.02832	0.00140	0.00892
17	0.01870	0.02778	0.14269	0.20612

NOTES: [a]Sectoral output is defined here in pure product terms (i.e., the "primary" products of a given sector, plus the "secondary" products of all other industries that are primary to the sector in question. For further definitional details, refer to Chapter II.

[b]Intermediate demand refers to the interindustry shipments of a homogeneous product group, domestically produced, regardless of which industry has actually produced it.

[c]Prediction error is measured by θ_i, where

θ_i: logarithmic prediction error for sector i, using natural logarithms.

z_i: actual domestic output of (or intermediate demand for) product class i ($i = 1, \ldots, n$) in 1961, expressed in constant 1958 dollars.

$\hat{z}_i$: predicted domestic output of (or intermediate demand for) product class i ($i = 1, \ldots, n$) in 1961, expressed in constant 1958 dollars.

of individual sectors, although the measure itself is used in assessing overall predictive performance. The detailed logarithmic prediction errors are used in analyzing the structure of prediction errors.

All of the overall measures of predictive performance are used in analyzing the comparative predictive performance of the two models at each level of aggregation, as well as at all levels of aggregation taken together. Attention is focused primarily on the observed results given in Table IV-5, referring to intermediate demand predictions, since, in the final analysis, input-output systems should be judged in terms of their prediction of levels of intersectoral demand for goods and services, given exogenously determined *bill of goods* for final consumption. Among the predictive measures, weighted mean logarithmic prediction error (θ^*) and weighted root mean square prediction error (RMS*) have been used in the actual statistical tests reported in the next part of this chapter.

It will be seen in Tables IV-5 and IV-6 that there are eight observations for each measure of overall predictive performance, two at each aggregation level for the two models. Thus, each observation can be seen as a function of the type of model, as well as of the aggregation level:

(4.61)
$$\gamma_k = f(m_r, a_s) \qquad k = 1,2,3,4,5,6; \quad r = 1,2; \quad s = 1,2,3,4;$$

where

γ_k is a particular observation, such as e, θ, θ^*, etc.;

m_r is the type of model, where the subscript r can take only two values, 1 or 2, denoting the *commodity technology* and the *industry technology* model, respectively; and,

a_s is the aggregation level, where the subscript s can take four values, 1,2,3,4, denoting the 79 x 79, 60 x 60, 45 x 45, and 17 x 17–order model formulations, respectively.

The first question we want to tackle is how to segment these observations, or how to cluster them, so that they can be studied systematically. Let us take the observations on θ^* or RMS*, for example, see if we can develop a criterion for clustering them. Suppose we use the criterion that the observations should be segmented in such a way that the various resulting clusters each display minimum variance. An inspection of Figures IV-1 through IV-4 should indicate visually that fitting a polynomial through the observation points for a given model would indeed lead to minimum variance. Other clustering possibilities suggest themselves, but can be discarded, since the overwhelming evidence is in favor of clustering the observations by model type.

The second important question that needs to be settled concerns the particular set of assumptions under which these results can be analyzed. What type assumptions, if any, can be made on the structure of measurement errors inherent in the two models and to what extent must the conclusions be qualified under such assumptions?

As pointed out in Chapter I, input-output observations are based on a single sample, yielding only zero degrees of freedom, and that although there is a best single estimate of a given parameter, there is no way of estimating the reliability of that estimate. Thus, if a given observation is a random drawing from a normally distributed population and we wish to estimate the mean of the distribution, the observation itself certainly represents an appropriate estimate of the mean, even though the variance of the distribution, which is the same as the variance of the estimate about the true mean, cannot be estimated.

The results of the experiments will be analyzed in the next part of this chapter, without making any particular set of assumptions on the structure of input-output measurement errors. Following such an analysis, the conclusions that are reached will be interpreted within the context of the fact that the variance of distribution of the population from which a given parameter is a random drawing (i.e., the variance of the estimate about the true mean) is unknown and cannot be estimated. This remark already suggests the exercise of considerable caution in drawing strong conclusions from the input-output experiments reported here, at least on the question of which of the two input-output models is *better.*

E. ANALYSIS OF RESULTS

The analysis of the results obtained from the input-output experiments will be conducted in three parts. In the first part, the comparative overall predictive performance of the *commodity technology* and *industry technology* models will be studied at each level of sectoral aggregation, as well as at all levels of aggregation taken together. After conducting the analysis without making any assumptions on the error structure of the models, the conclusions will be interpreted within the context of the presence of measurement errors in input-output models. In the second part, attention will be given to the relationship between the aggregation level of the models and overall measures of predictive performance. In the third part, the detailed structure of the input-output prediction errors will be studied, not only to explain in some depth the underlying reasons for observed measures of overall predictive performance but also to test for the existence of significant relationships between the detailed prediction errors and a number of variables.

1. The Comparative Predictive Performance of the "Commodity Technology" and "Industry Technology" Models.

An analysis of the comparative predictive performance of the two models can be performed at each level of aggregation separately, as well as at all levels of aggregation taken together. First, some general observations will be made on the comparative predictive performance of the two models by considering the various measures of predictive performance used in the experiments, such as the Weighted Mean Logarithmic Prediction Error (θ^*) and the Weighted Root Mean Square Prediction Error (RMS*). Secondly, the predictive performance of the two models will be tested for significant differences. Lastly, the conclusions will be interpreted within the context of measurement errors inherent in the models used.

(a) General Observations

Observation 1:

An examination of Tables IV-5 and IV-6, as well as Figures IV-1 through IV-4 indicates that the *industry technology* model scores consistently better (i.e., over all aggregations) than the *commodity technology* model in predicting intermediate demand levels, in terms of Mean Relative Prediction Error (e), Root Mean Square Prediction Error (RMS), Weighted Root Mean Square Prediction Error (RMS*), and Pearsonian Coefficient of Correlation (r).

The same conclusions hold true when output predictions are considered, with the only (and minor) exception that at the 17 x 17 level of aggregation, r is perfect (i.e., 1.00000) for the *commodity technology* model, whereas it is slightly less perfect (i.e., 0.97918) for the *industry technology* model.

Observation 2:

When the comparisons are made in terms of the Mean Logarithmic Prediction Error (θ) or the Weighted Mean Logarithmic Prediction Error (θ^*), the results are not as clearcut. In terms of θ, the *commodity technology* model maintains its superiority at aggregation levels I, II, and III, and gives inferior results only at aggregation level IV (i.e., 17 x 17 level). However, in terms of θ^*, the superiority of the *commodity technology* model ends at aggregation level III.

Observation 3:

Depending upon the particular measure used, the superiority of either the *commodity technology* or the *industry technology* model at different levels of aggregation shows some important variations. RMS and RMS* for the two models diverge, for example, as the models become more and more aggregated, such that the superiority of the *industry technology* model becomes very substantial at aggregation level IV. On the other hand, if we examine the behavior of e, we see that the two models are closer together at aggregation levels II and III, while they diverge substantially at aggregation levels I and IV (i.e., at the most detailed and the most aggregated levels). Further, when attention is shifted to a comparative analysis of the *goodness of fit* for intermediate demand predictions, it will be seen that while the *industry technology* model is consistently superior to the *commodity technology* model this superiority is negligible at aggregation levels I and II and becomes more clearly evident at aggregation levels III and IV.

Observation 4:

An examination of these results leads to the preliminary conclusion that it is difficult to judge one model to be superior to the other at a given level of aggregation, as they seem to possess particular strengths and weaknesses at different aggregation levels, depending on the measures of predictive performance that are used. It would seem that if a researcher is interested in minimizing geometric mean of the prediction errors at the 79 x 79 level of sectoral aggregation, he would probably prefer to use the *commodity technology* model. If, on the other hand, his interest lies in minimizing the absolute prediction error at the same level of aggregation (i.e., 79 x 79), he would prefer the *industry technology* model. Similarly, he probably would be indifferent at the 50 x 50 or 55 x 55 level as to which of the two models he should use, if he is primarily concerned with the geometric mean of the prediction errors (note the point of intersection of the two curves in Figures IV-3 and IV-4).

The superiority of either model at *all levels of aggregation taken together* is similarly difficult to determine. However, as already noted, the *industry technology* model performs consistently better, generally speaking, except for the fact this superiority is not clearly evident when the two models are compared in terms of the measures θ and θ^*. If it can be established through a statistical test that the *industry technology* model is significantly different from the *commodity technology* model in terms of θ and θ^*, it can be concluded that the *industry technology* model possesses overall superiority over all aggregation levels taken together. Still, such a conclusion will fail to help a researcher in deciding which model to use at a given level of aggregation.

(b) Tests for Significant Differences between the *Commodity Technology* and *Industry Technology* Models at All Levels of Aggregation.

It was observed above that in order to conclude that the *industry technology* model possesses overall superiority over all aggregation levels taken together, it would be necessary to conduct statistical tests using the predictive measures θ and θ^*, showing that there exists a significant difference between the two models. First, it would seem that for such a test the measure θ^* is a more meaningful index to use, since it is a weighted index and would seem to give a better measure of the location (mean) of logarithmic prediction errors. Also, it should suffice to test the null hypothesis,using θ^*, that the mean prediction error of the *commodity technology* model is equal to that of the *industry technology* model, against the alternative hypothesis that the mean prediction error of the *commodity technology* model is greater than that of the *industry technology* model. The formulation of the alternative hypothesis in this way would be particularly well suited to the problem, since the *commodity technology* model possesses only a slight edge over the *industry technology* model at the 79 x 79 and 60 x 60 levels of aggregation, while the *industry technology* model yields substantially lower values of θ^* at the 45 x 45 and 17 x 17 levels of aggregation.

Before proceeding with a t-test for difference of two means, it is necessary to recall the latter part of the theoretical discussion given earlier in this chapter on the structure of prediction errors. Specifically, concern was expressed for the stochastic independence of the observed measures of overall predictive performance (i.e., the independence of the respective values of θ^* for either of the two models). Generally, if there is reason to suspect that the observations may not be randomly and independently distributed, then it is necessary to test the randomness of the successive observations before the usual statistical methods based on randomness can be applied.

Among available methods that can be used in testing for randomness, the test based on runs, as pointed out by Hoel, is a poor test in many respects.[31] We can use the mean square successive difference test, which is a test of the null hypothesis that we have a sequence of independent observations θ_1^*, $\theta_2^*,\ldots,\theta_n^*$ from a population $N(\mu,\sigma)^2$. A description of this method can be found in Brownlee,[32] von Neumann, *et al.*,[33] and Hart.[34]

The Mean Square Successive Difference Test for Randomness

In applying the mean successive difference test, we compute σ^2 in two ways. The first is the unbiased estimator.[35]

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[31] For example, it is not likely to discover certain types of nonrandomness of a cyclical nature unless the observations are spaced just right. Further, there are many types of nonrandomness that may occur but will go undetected by the runs test because the total number of runs is roughly equal to the number expected for a random sequence. Refer to Hoel, *op. cit.,* pp. 341-342.

[32] Brownlee, *op. cit.,* pp. 221-223.

[33] J. von Neumann, R.H. Kent, H.R. Bellinson and B.I. Hart, "The Mean Square Successive Difference," *The Annals of Mathematical Statistics,* XII (1941), 153-162.

[34] B.I. Hart, "Significance Levels for the Ratio of the Mean Square Successive Difference to the Variance," *The Annals of Mathematical Statistics,* XIII (1942), 445-447.

[35] In computing this, it is usually convenient to use the identity

$$\sum_{i=1}^{n} (x_i-\bar{x})^2 = \sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + n\bar{x}^2$$

$$= \sum_{i=1}^{n} x_i^2 - 2\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)\sum_{i=1}^{n} x_i + n\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)^2$$

$$= \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2$$

See Brownlee, *op. cit.,* p. 222.

(4.62)

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

The second estimator of σ^2 is $d^2/2$, where d^2 is defined as

(4.63)

$$d^2 = \frac{1}{n-1} \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 .$$

It is easy to see that $E[d^2/2] = \sigma^2$, since[36]

(4.64)

$$E[d^2] = \frac{1}{n-1} E\left[\sum_{i=1}^{n-1} x_{i+1}^2 + \sum_{i=1}^{n-1} x_i^2 - 2\sum_{i=1}^{n-1} x_{i+1} x_i\right]$$

$$= \frac{1}{n-1} \left\{ \sum_{i=1}^{n-1} E[x_{i+1}^2] + \sum_{i=1}^{n-1} E[x_i^2] - 2\sum_{i=1}^{n-1} E[x_{i+1}] E[x_i] \right\}$$

$$= 2\left\{E[x_i^2] - (E[x_i]^2)\right\} = 2V[x] = 2\sigma^2 .$$

It was proved by von Neumann, *et al.*,[37] that under the null hypothesis,

(4.65)

$$E\left[\frac{d^2/2}{s^2}\right] = 1,$$

and

(4.66)

$$V\left[\frac{d^2/2}{s^2}\right] = \frac{n-2}{n^2-1} .$$

Thus the test statistic

(4.67)

$$u_p = \frac{\frac{d^2/2}{s^2} - 1}{\sqrt{(n-2)/(n^2-1)}}$$

is approximately distributed as a unit normal deviate under the null hypothesis.

---

[36] *Ibid.*

[37] von Neumann, *et al., loc cit.*

The alternative to the null hypothesis is usually that the consecutive observations tend to be correlated positively with their predecessors. The successive differences $x_{i+1} - x_i$ therefore tend to be smaller under the alternative hypothesis than they would be under complete randomness, so the expected value of $d^2/2$ is less than σ^2 and the numerator of (4.67) will tend to be negative.

Using the observations on θ_1^*, θ_2^*, θ_3^*, and θ_4^* for the *commodity technology* model for intermediate demand predictions (Table IV-5), we can compute s^2 to be 0.0108243, with $\bar{\theta}^* = 0.0943900$ and $n=4$.

Similarly, by ordering the θ^* values by level of sectoral aggregation k, from most to least detailed,

$$
\begin{aligned}
\theta^*_{k=1} &= 0.02057 \\
\theta^*_{k=2} &= 0.02068 \\
\theta^*_{k=3} &= 0.09504 \\
\theta^*_{k=4} &= 0.24129
\end{aligned}
\tag{4.68}
$$

We can compute d^2 as follows:

$$
\begin{aligned}
d^2 &= \frac{1}{3} \sum_{k=1}^{3} (0.00011)^2 + (0.07436)^2 + (0.14625)^2 \\
&= 0.0089728.
\end{aligned}
\tag{4.69}
$$

As a result we can determine that

$$
\frac{d^2/2}{s^2} = \frac{0.0044864}{0.0108243} = 0.4144757 \tag{4.70}
$$

so that our test statistic is

$$
u_p = \frac{0.5855243}{\sqrt{0.1333333}} = -1.6035168 \tag{4.71}
$$

which leads to a p-value[38] of p = 0.0548 at $\alpha = 0.005$ level of significance, which then leads us to the conclusion that since the p-value is substantially greater than $\alpha = 0.005$, we would not reject the null hypothesis of randomness at the $\alpha = 0.005$ level of significance. This would seem to indicate that there is less than one chance in 100 that we would reject the null hypothesis of randomness when it should be accepted. That is, the probability of committing a Type I error (i.e., rejecting the null hypothesis when it should be accepted) is extremely small.

When the statistical testing process outlined here is repeated, this time using the observations on $\theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*$ for the *industry technology* model and again for intermediate demand predictions, we find the following numerical results:

$$
\begin{aligned}
s^2 &= 0.0009439 \\
d^2 &= 0.0006848 \qquad (4.72) \\
u_p &= 0.0405,
\end{aligned}
$$

where the test statistic u_p leads to the same conclusion.

Of course, in applying such a test, caution must be exercised so as not to commit a Type II error (i.e., accepting the hypothesis of randomness when in fact it should be rejected). The conventional procedure of choosing α and leaving β (i.e., the size of Type II error to fend for itself often appears as an arbitrary asymmetry. On the other hand, the acceptance of the null hypothesis does not imply a belief that it is true, merely that the evidence of rejecting it is insufficient. Unfortunately, the hypothesis testing procedure itself does not distinguish between the case in which the null hypothesis is either true or close to being true and the case in which the data are simply insufficient to draw any useful conclusion.

---

[38] The p-value can be easily found by using a table showing the cumulative standardized normal distribution function, as given by Brownlee, *op. cit.,* Table I, pp. 558-559. The table given by Brownlee is an abridged version of Table II given in A. Hald, *Statistical Tables and Formulas* (New York: John Wiley and Sons, Inc., 1952).

t-test for Difference of Two Means[39]

We can now conduct a test to see if the mean prediction error for the *commodity technology* model is significantly different from that of the *industry technology* model, when such a comparison is made over all aggregation levels taken together. Employing the observations on θ^* obtained for the two models in predicting intermediate demand levels to 1961, we conduct the test as follows.

Let the observations θ^* coming from the *commodity technology* model be denoted by x and those coming from the *industry technology* model be denoted by y. Let x and y be normally distributed with means μ_x and μ_y and with the same variance σ^2. Further, let random samples of sizes n_x and n_y be taken from these two populations. We can denote the sample means and variances by $\bar{x}$, $\bar{y}$, s_x^2, and s_y^2. Then

$$(4.73) \qquad u = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{\sigma_{\bar{x} - \bar{y}}} = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{\sigma \cdot \sqrt{\frac{1}{n_x} + \frac{1}{n_y}}}$$

will approach the standard normal distribution as the sample size increases. Furthermore,

$$(4.74) \qquad v^2 = \frac{n_x s_x^2 + n_y s_y^2}{\sigma^2}$$

with

$$(4.75) \qquad v = n_x + n_y - 2$$

degrees of freedom.

---

[39] Hoel, *op. cit.*, pp. 276-279.

It has been shown that

$$(4.76) \qquad \frac{n_x s_x^2}{\sigma^2} \quad \text{and} \quad \frac{n_y s_y^2}{\sigma^2}$$

possess independent χ^2 distributions with $n_x - 1$ and $n_y - 1$ degrees of freedom, respectively. Consequently,

$$(4.77) \qquad t = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{\sqrt{n_x s_x^2 + n_y s_y^2}} \cdot \sqrt{\frac{n_x n_y (n_x + n_y - 2)}{n_x + n_y}},$$

$$v = n_x + n_y - 2$$

will have Student's t distribution with $n_x + n_y - 2$ degrees of freedom.

Then, to test the null hypothesis that

$$(4.78) \qquad H_0 : \mu_x = \mu_y$$

it is merely necessary to calculate the value of t and use a table showing Student's t distribution to see whether the sample value of t numerically exceeds the critical value.

The numerical results in applying the test can be summarized as follows:

$$(4.79) \qquad \begin{aligned} \bar{x} &= 0.09439, & s_x^2 &= 0.01082430 \\ \bar{y} &= 0.05908, & s_y^2 &= 0.00094385 \\ & & n_x &= 4 \\ & & n_y &= 4 \end{aligned}$$

(4.80)

$$t = \frac{(0.09439 - 0.05908) - (\mu_x - \mu_y)}{\sqrt{(4)(0.01082430) + (4)(0.00094385)}} \cdot \sqrt{\frac{(4)(4)(4+4-2)}{4+4}}$$

$$= 0.558851$$

with 6 degrees of freedom. At $\alpha = 0.05$ level of significance, the critical t value is 1.943.

Since t = 0.558851 is substantially less than t = 1.943, we will not reject the null hypothesis of no difference between the mean prediction errors for the two models at $\alpha = 0.05$ level of significance.

We can continue this analysis by assuming unequal variances, since the preceding application of the test was valid only under the assumption that $\sigma_x = \sigma_y$. If $\sigma_x \neq \sigma_y$, but the values of σ_x and σ_y are known, one can test the hypothesis $\mu_x = \mu_y$ by means of the standard normal variable

(4.81) $$\tau = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{\sigma_{\bar{x} - \bar{y}}}$$

$$= \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{\sqrt{\dfrac{\sigma_x^2}{n_x} + \dfrac{\sigma_y^2}{n_y}}}$$

The values of the two variances are seldom known; therefore it is usually necessary to replace them by their sample estimates.

If σ_x^2 and σ_y^2 are replaced by their unbiased sample estimates,

(4.82) $$\hat{\sigma}_x^2 = \frac{\sum_{i=1}^{n_x} (x_i - \bar{x})^2}{n_x - 1} \quad \text{and} \quad \hat{\sigma}_y^2 = \frac{\sum_{i=1}^{n_y} (y_i - \bar{y})^2}{n_y - 1}$$

the resulting variable

(4.83) $$t = \frac{(\bar{x} - \bar{y}) - (\mu_x - \mu_y)}{\sqrt{\dfrac{\hat{\sigma}_x^2}{n_x} + \dfrac{\hat{\sigma}_y^2}{n_y}}}$$

can be shown to possess an approximate Student t distribution. This is not surprising, given the fact that Student's t is obtained by replacing the unknown variance by its unbiased sample estimate in the corresponding expression for a single variable. The number of degrees of freedom necessary to make (4.83) an approximate t variable is given by a rather elaborate formula, namely,

(4.84)
$$v = \frac{\left(\frac{\hat{\sigma}_x^2}{n_x} + \frac{\hat{\sigma}_y^2}{n_y}\right)^2}{\frac{\left(\frac{\hat{\sigma}_x^2}{n_x}\right)^2}{n_x+1} + \frac{\left(\frac{\hat{\sigma}_y^2}{n_y}\right)^2}{n_y+1}} - 2 \; .$$

Again, summarizing our data as

(4.85)
$$\bar{x} = 0.09439 \, , \qquad \hat{\sigma}_x^2 = 0.01082430$$
$$\bar{y} = 0.05908 \, , \qquad \hat{\sigma}_y^2 = 0.00094385$$
$$n_x = n_y = 4$$

we can compute the t and v values, using Eqs. (4.83) and (4.84):

(4.86)
$$t = 0.65100$$
$$v = 5.9 - 2 \approx 4 \text{ degrees of freedom}$$

With 4 degrees of freedom, the critical t value at $\alpha = 0.05$ level of significance is 2.132, which is substantially higher than $t = 0.6500$. Consequently we *again cannot* reject the null hypothesis of equal means in overall prediction errors at $\alpha = 0.05$ level of significance.

A number of other, nonparametric methods can also be used to test the hypothesis of equal means in overall prediction errors, such as the Wilcoxon Two-Sample Rank Test, that can be applied relatively easily and that do not require the normality assumption on population distributions.[40]

(c) Interpretation of the Results on the Comparative Predictive Performance of the Two Models in View of the Presence of Measurement Errors

In view of the fact that the error structure of the two models is unknown, the conclusions reached earlier on the comparative predictive performance of the two models must be taken with extreme caution. When we examine the diagrams given in Figures IV-1 through IV-4, we should note that whether or not the two observations for the two models at a given aggregation level is *significantly different* is pretty much conditioned by the respective errors underlying these observations. Similarly, when we wish to make

---

[40] For an excellent source on nonparametric methods, emphasizing detailed interpretations rather than a *how-to-do-it* approach, refer to James V. Bradley, *Distribution-Free Statistical Tests* (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1968).

the assessment that one function is substantially different from the other, here again considerable care must be exercised, since the error bounds on the two functions (curves) is simply unknown.

This should not suggest, however, that since the error structure of input-output systems are unknown they are not of much use or that we are unable to say anything significant on their empirical properties. It would seem, quite to the contrary, that through sensitivity experiments, such as those essentially presented here, we can learn sustantially on the properties of these models. Since a considerable amount of demand is now put on input-output models for multisectoral economic forecasting or for economic development programming, some of the conclusions reached here should be of some help in deciding what type of model to use and how it compares with other alternatives in a number of respects.

2. The Relationship between Sectoral Aggregation Levels and Selected Measures of Overall Predictive Performance

The relationship between the RMS* (Weighted Root Mean Square Prediction Error) and θ^* (Weighted Mean Logarithmic Prediction Error) and the aggregation levels is presented in Tables IV-1 through IV-4, for both the *commodity technology* and the *industry technology* models, for intermediate demand as well as for output predictions. The shape of the functions in these diagrams clearly indicates a nonlinear relationship between the overall prediction error used and the aggregation level inherent in the model, such that the relationship can be accurately described by fitting a third degree polynomial through the observation points.

When the analysis is conducted in terms of RMS*, it can be seen that the value of RMS* rises rapidly with an increase in the aggregation level (i.e., moving from the most detailed to the least detailed model). This observation holds true for both models, and for the intermediate demand and output predictions. If, on the other hand, the analysis is conducted in terms of θ^*, then the two models give somewhat different overall results. While the *industry technology* model shows a fairly stable and still slightly nonlinear relationship, the *commodity technology* model shows a definite nonlinear relationship in which the overall prediction error rises very steeply as the model becomes more and aggregated.

FIGURE IV-1

THE RELATIONSHIP BETWEEN WEIGHTED ROOT MEAN SQUARE PREDICTION ERRORS AND LEVELS OF SECTORAL AGGREGATION IN THE "COMMODITY TECHNOLOGY" AND "INDUSTRY TECHNOLOGY" MODELS

(Output Predictions, Competitive Imports Treated Exogenously)

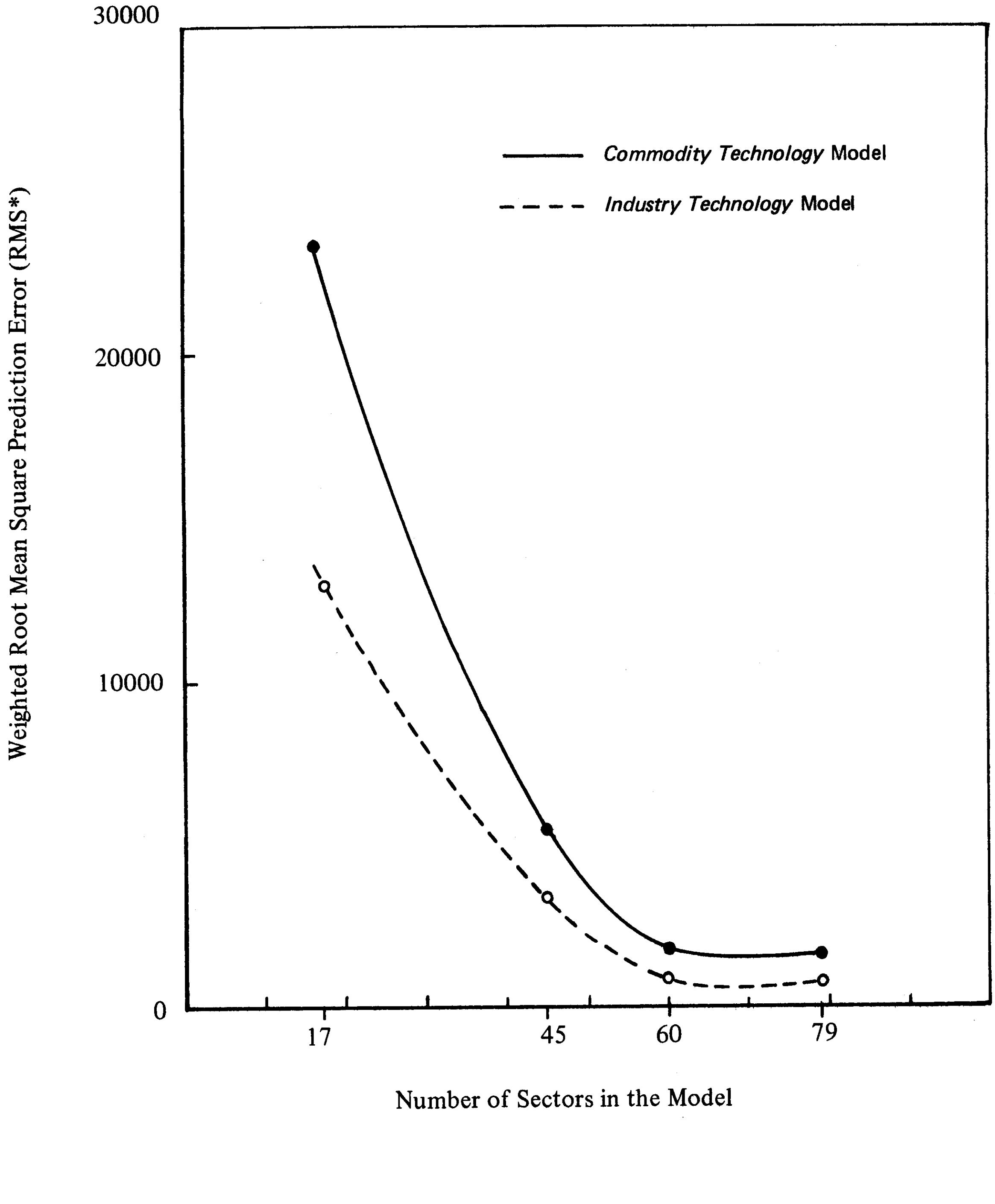

FIGURE IV-2

THE RELATIONSHIP BETWEEN WEIGHTED ROOT MEAN SQUARE PREDICTION ERRORS AND LEVELS OF SECTORAL AGGREGATION IN THE "COMMODITY TECHNOLOGY" AND "INDUSTRY TECHNOLOGY" MODELS

(Intermediate Demand Predictions,
Competitive Imports Treated Exogenously)

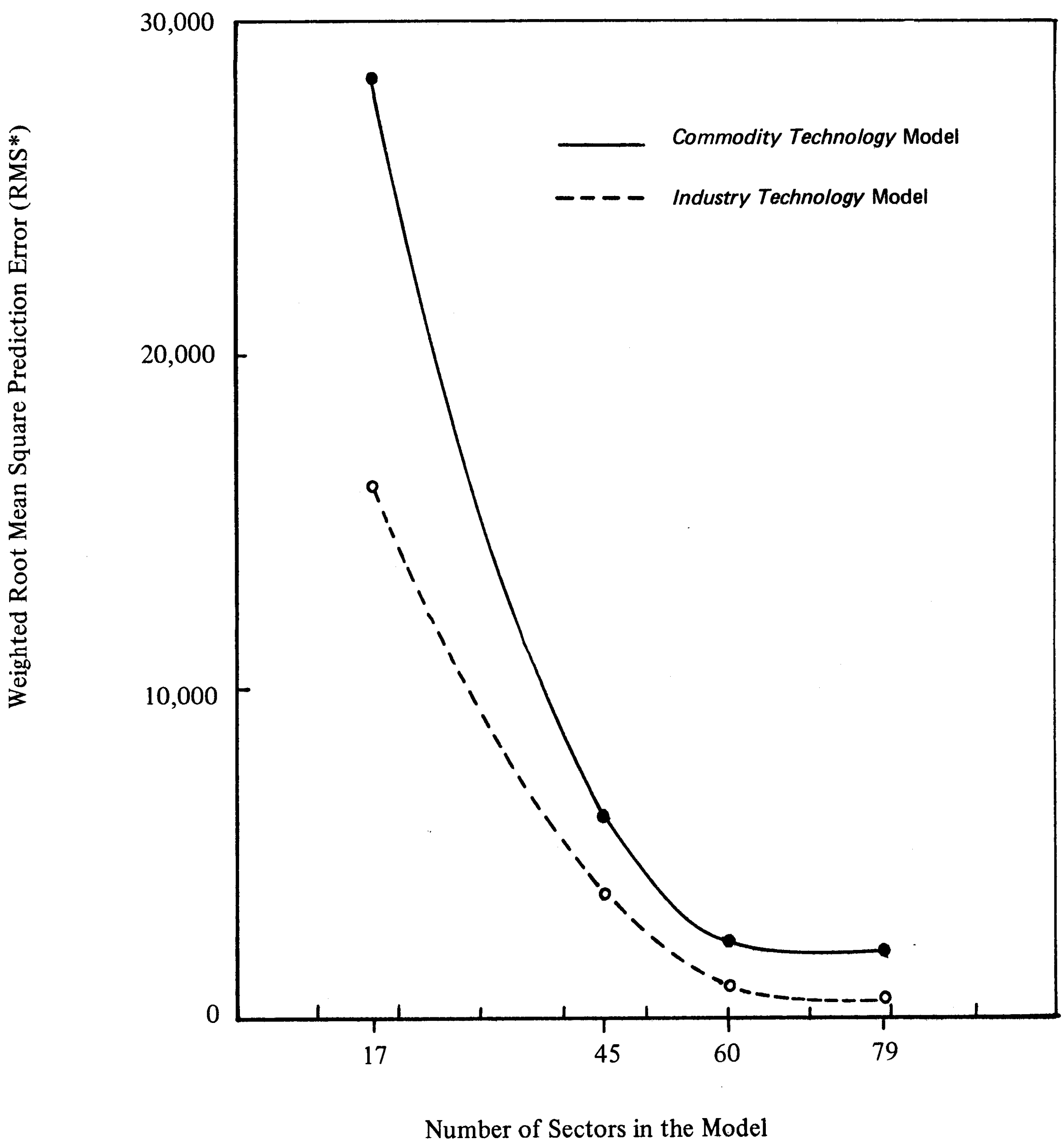

FIGURE IV-3

THE RELATIONSHIP BETWEEN WEIGHTED MEAN LOGARITHMIC PREDICTION ERRORS AND LEVELS OF SECTORAL AGGREGATION IN THE "COMMODITY TECHNOLOGY" AND "INDUSTRY TECHNOLOGY" MODELS

(Intermediate Demand Predictions,
Competitive Imports Treated Exogenously)

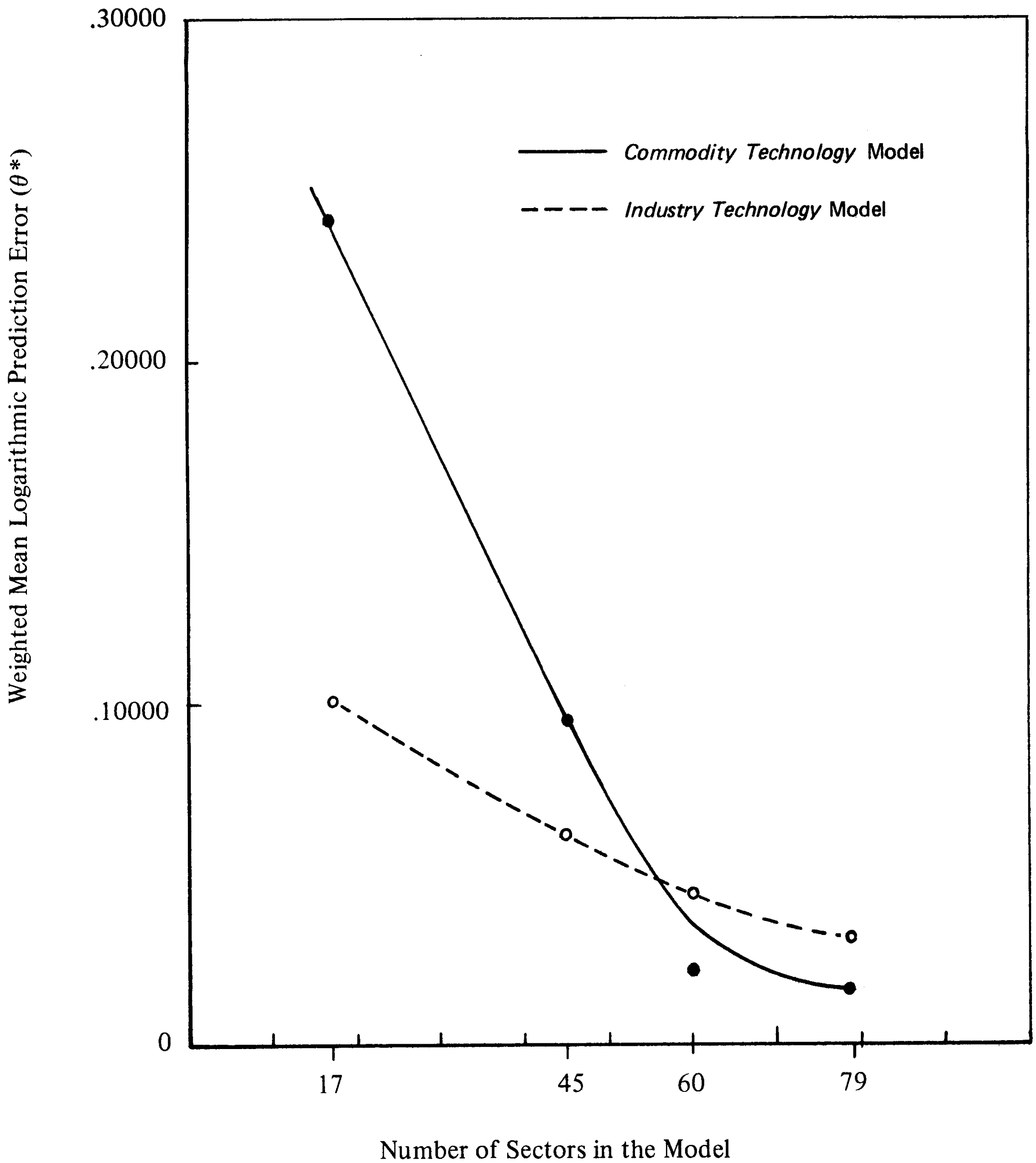

FIGURE IV-4

THE RELATIONSHIP BETWEEN WEIGHTED MEAN LOGARITHMIC PREDICTION ERRORS AND LEVELS OF SECTORAL AGGREGATION IN THE "COMMODITY TECHNOLOGY" AND "INDUSTRY TECHNOLOGY" MODELS

(Output Predictions, Competitive Imports Treated Exogenously)

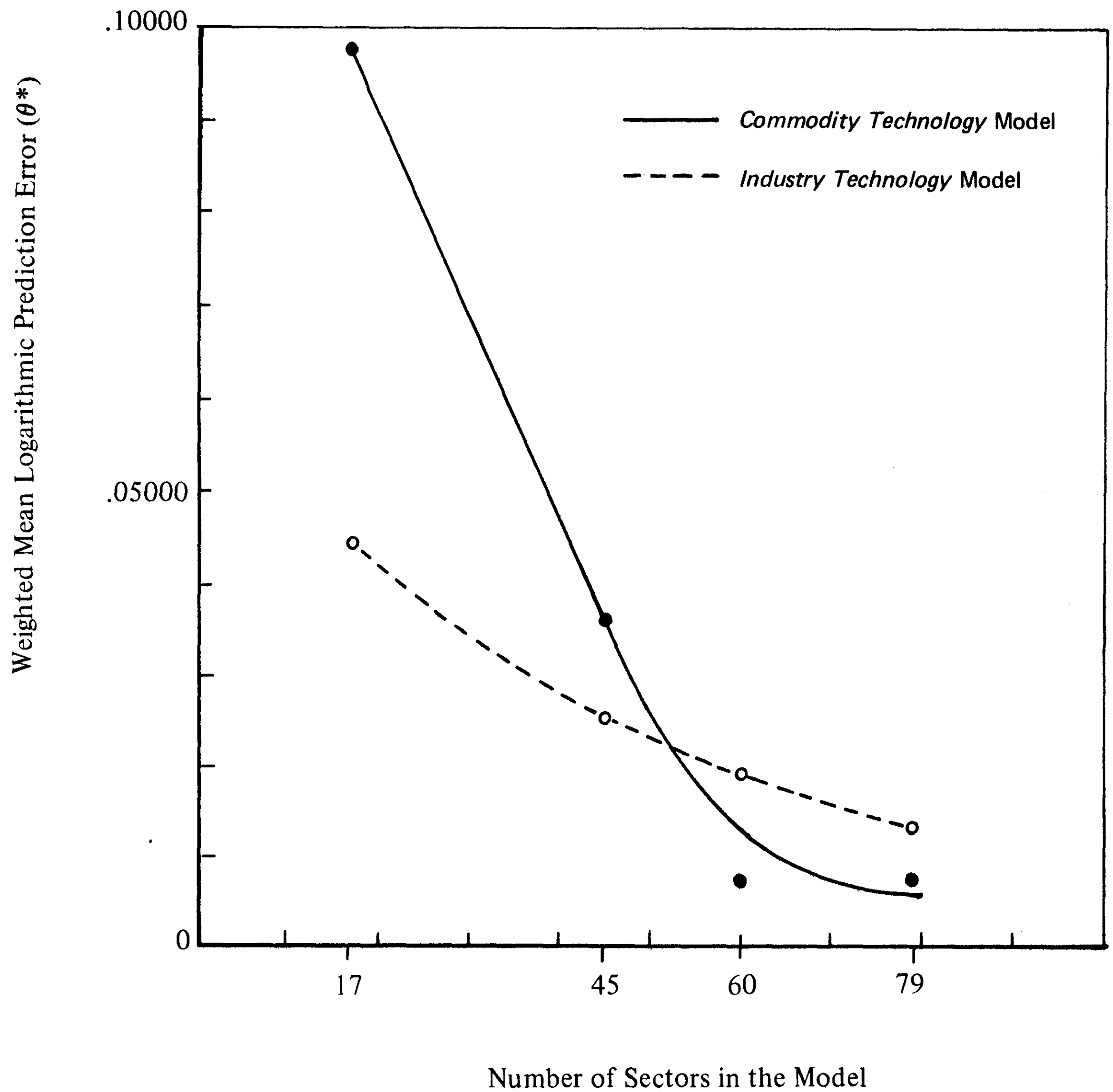

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FIGURE IV–5

PREDICTION–REALIZATION DIAGRAM,
INTERMEDIATE DEMAND PREDICTIONS TO 1961
AT THE 79 x 79 LEVEL OF SECTORAL AGGREGATION
(millions of constant 1958 dollars)

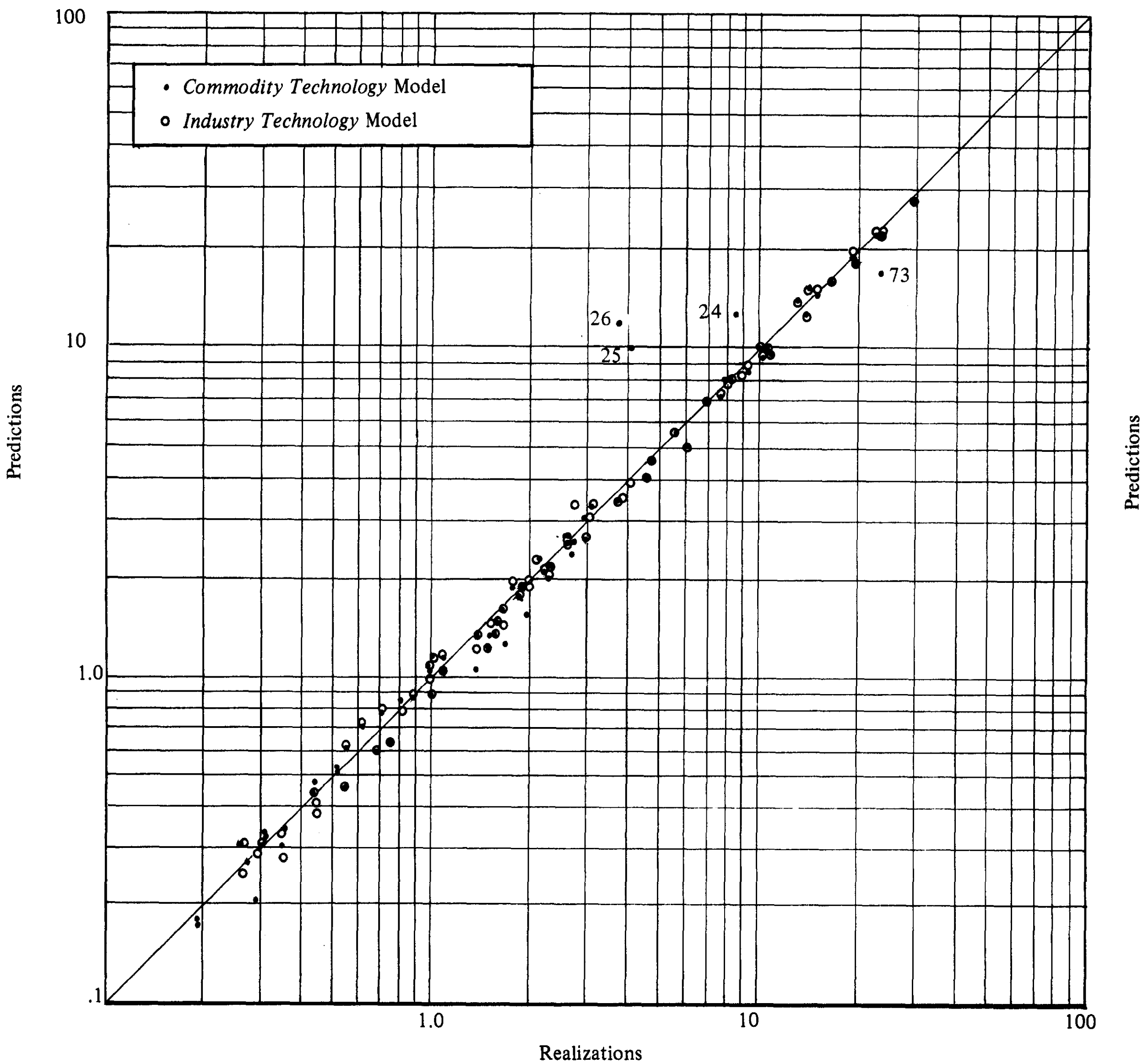
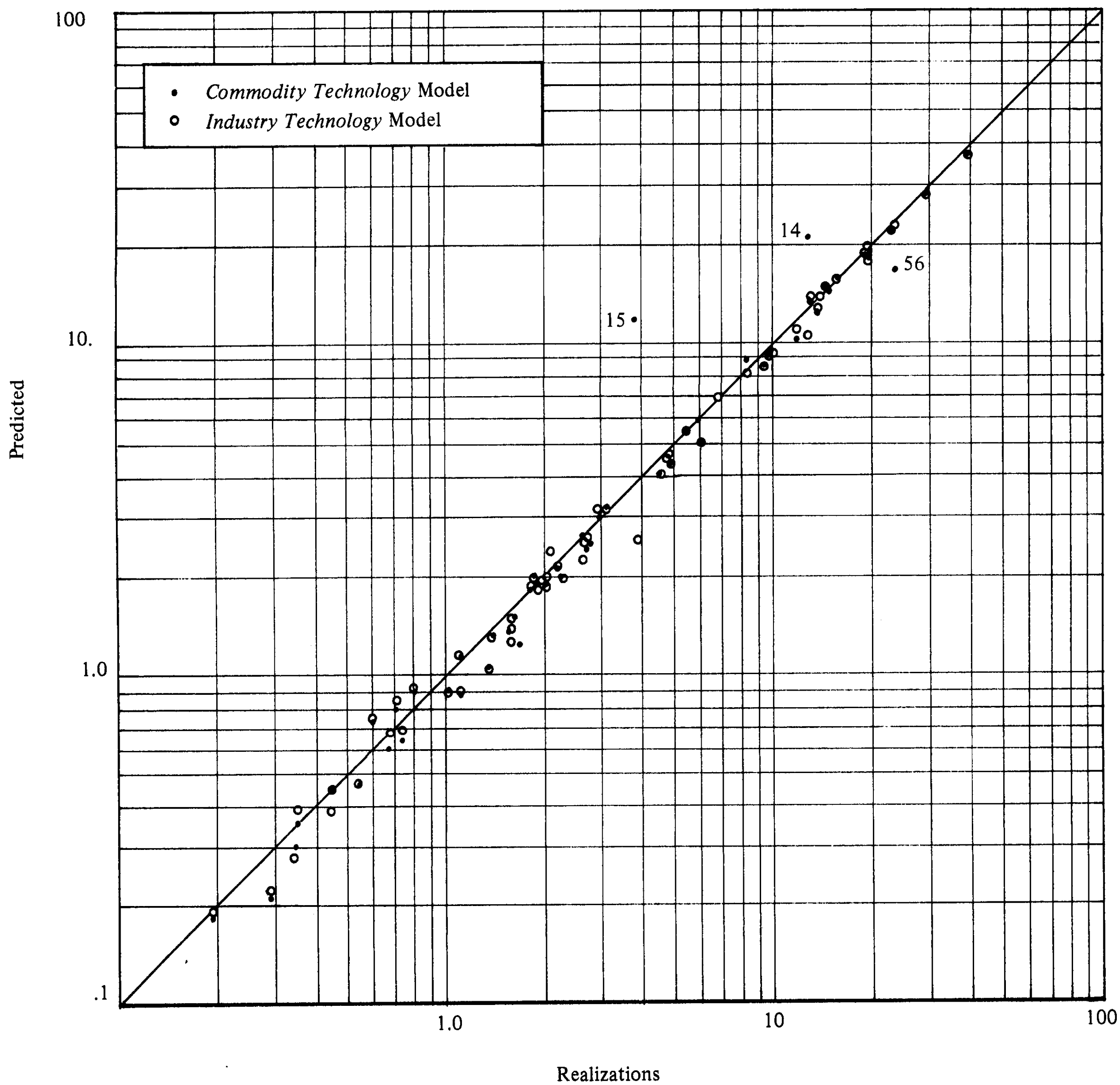


FIGURE IV–6

PREDICTION–REALIZATION DIAGRAM,
INTERMEDIATE DEMAND PREDICTIONS TO 1961
AT THE 60 x 60 LEVEL OF SECTORAL AGGREGATION
(millions of constant 1958 dollars)



Predictions

FIGURE IV–7

PREDICTION–REALIZATION DIAGRAM,
INTERMEDIATE DEMAND PREDICTIONS TO 1961
AT THE 45 x 45 LEVEL OF SECTORAL AGGREGATION
(millions of constant 1958 dollars)

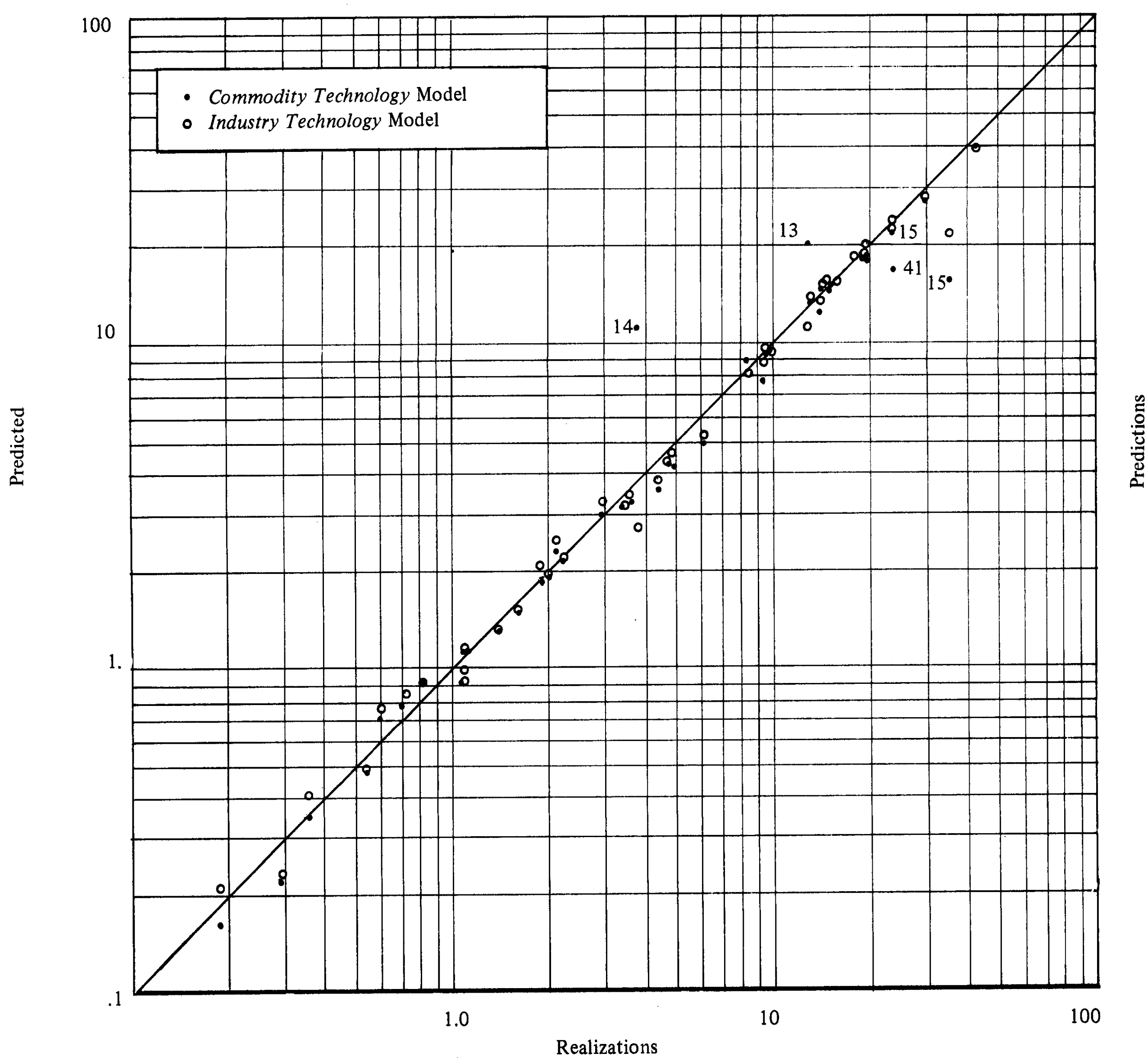
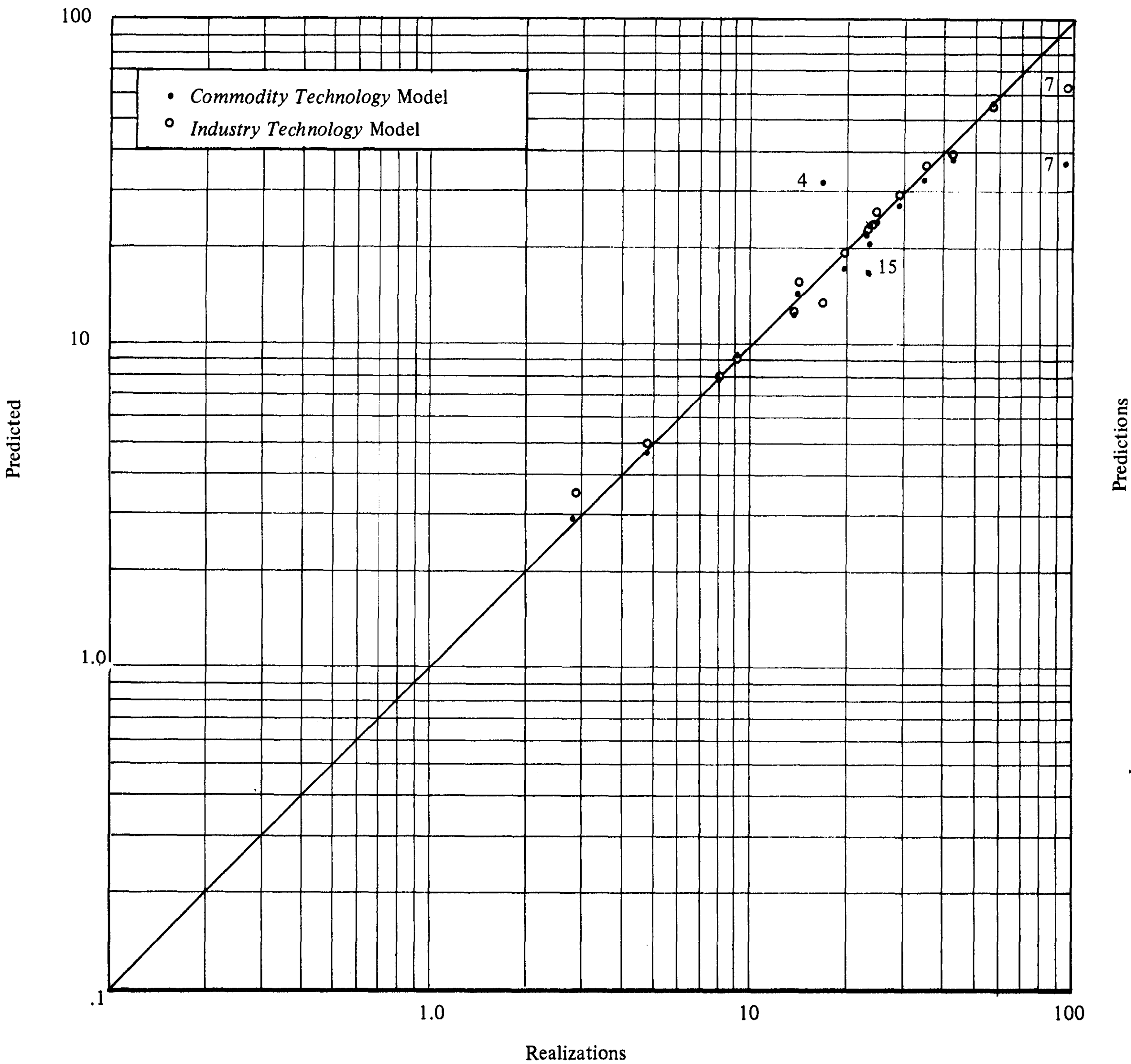


FIGURE IV–8

**PREDICTION–REALIZATION DIAGRAM,
INTERMEDIATE DEMAND PREDICTIONS TO 1961
AT THE 17 x 17 LEVEL OF SECTORAL AGGREGATION
(millions of constant 1958 dollars)**



Conducting statistical tests to see if there exists a significant association between the overall prediction errors and the sectoral aggregation level would not seem to be warranted, in view of the clearcut functional relationships suggested in Figures IV-1 through IV-4.

3. The Structure of Detailed Prediction Errors

The discussion here will be centered on the following two areas. First, some general observations will be made to explain some of the reasons underlying the overall predictive behavior of the two models, particularly the *commodity technology* model, by analyzing the structure of detailed prediction errors. Secondly, the results of a series of tests will be discussed, which were conducted to see if the absolute prediction errors displayed any systematic characteristics, and to see if logarithmic prediction errors showed a significant correlation with a number of *dimensional* variables.

a. Some Explanations of Overall Predictive Behavior by Examining Predictions for Individual Sectors

After having examined the overall prediction results, it is instructive to study the predictive performance of the two models at the level of individual sectors, at each level of sectoral aggregation. Such an analysis can be facilitated through the prediction-realization diagrams presented in Figures IV-5 through Figure IV-8.

A visual examination of these diagrams would indicate that most sectors are clustered tightly about the *line of perfect predictions* and that a few industries are significantly deviant with respect to this line. Those above the line indicate *over-predictions,* and others that lie under the line represent *under-predictions.* These diagrams refer specifically to the intermediate demand predictions. Output predictions could similarly be depicted in the same type of diagrams.

When the *deviant* sectors are investigated in terms of the type of model used, it will be seen that significant deviations occur at the 79 x 79 and 60 x 60 levels only for the *commodity technology* model, while the *industry technology* model yields very good results. At the 45 x 45 and 17 x 17 levels of aggregation, however, the *industry technology* model produces two significant deviations (both under-predictions), for sector 15 (Chemicals complex) at the 45 x 45 level and for sector 7 (Metals complex) at the 17 x 17 level. It can be seen that these same sectors are also under-predicted by the *commodity technology* model at the same levels of aggregation.

When the deviant sectors are examined in terms of their recurrence at different levels of aggregation, we see, for the *commodity technology* model, that the same set of sectors show up on each diagram. At the 79 x 79 level, we see that sectors 24, 25, and 26 (Paper Products, Printing and Publishing sectors) are over-predicted, and they, as a group, are over-predicted at all levels of aggregation. Similarly, sector 73 at the 79 x 79 level (Business Services) is systematically under-predicted.

It would seem that since the overall results did not much favor the *commodity technology* system, a somewhat more detailed study of why these individual sectors give *bad* predictions is warranted. This can be done by going back to the original information used in developing the *commodity technology* model, to see if any errors might have been introduced during the data processing stages of these experiments or, alternatively, if the procedures used in compiling the basic information underlying the model might have caused these results.

An analysis of this problem indicates that in the absence of any glaring mistakes during the data processing stages of the experiments, we should look at the procedures used in compiling the basic intersectoral flows information used in developing the *commodity technology* model. The fundamental procedural fact responsible for these results has been uncovered and can be explained as follows.

An examination of the *make* matrix given in Appendix F should indicate that both the Printing and Publishing industry (Sector 26) and the Radio and TV Broadcasting industry (Sector 67) produce secondary products, consisting mostly of advertising services, that are primarily produced by the Business Services sector (Sector 73). In fact, the Printing and Publishing industry's output of advertising services was well over five billion dollars in 1958, while that of the Radio and TV Broadcasting industry was about one and a half billion dollars. Ideally, in setting up the basic intersectoral commodity flows matrix, these advertising services should have been lifted out of the Printing and Publishing, as well as the Radio and TV Broadcasting sectors and classified under the Business Services sector for distribution to all the consuming sectors. This procedure was not followed by the National Planning Association, from which the basic intersectorsl flows information underlying the *commodity technology* model was obtained. Under the NPA procedure, the advertising output of these two industries was considered a primary output of these two sectors, rather than that of the

Business Services sector, and such output was distributed along the respective rows to their purchasers. As a consequence, the Transportation sector (69) was shown as having purchased over a dollars worth of printing and publishing services, while discounting its advertising component, this would have amounted to about $175 million as recorded in the *commodity-to-industry flows* matrix presented in Appendix F. Many other sectors were similarly shown as purchasing printing and publishing services or radio and TV broadcasting services, when in fact they were purchasing advertising services rendered by industries other than the Business Services industry. This practice led to larger *technological* coefficients in the *commodity technology* matrix and further in the Leontief inverse, causing over-predictions for the Printing and Publishing sector. Conversely, the same procedure led to the specification of smaller coefficients in the Business Services row and this, in turn, resulted in under-predictions for this sector. The under-prediction for the Paper and Paper Products sector and for the Paper Containers sector the 79 x 79 level of aggregation is somewhat more difficult to explain. A similar problem as discussed with respect to the Printing and Publishing and the Radio and TV Broadcasting sectors should be suspected.

The observations given here should make it very clear that one must be extremely cautious in judging one model to be superior to the other, especially when we see, as discussed here, that these models are so sensitive in their predictive performance to the empirical specification of a few entries in the underlying intersectoral interdependence matrices.

b. An Analysis of Systematic Variations in the Prediction Errors

In order to see if the prediction errors showed any systematic variations, two sets of tests were conducted.

In the first set of tests, the absolute values of the intermediate demand prediction errors were related to the observed 1961 intermediate demand levels, for each aggregation level at a time, by using least-squares regression. In the second set of tests, and again using least-squares regression, the logarithmic intermediate demand prediction errors were related to a set of *dimensional* variables, as will be indicated below.

The results of the first set of tests are recorded in Tables IV-11 and IV-12. In these tests, the *deviant* sectors (refer to Figures IV-5 through IV-8) were omitted from the regression runs to see if, in their absence, a significant relationship would be observed between the absolute value of the intermediate demand prediction errors and the actual 1961 intermediate demand levels. Basically two types of conclusions emerge from these tests. First, the

Pages 207 and 208 do not exist. This is a pagination error by the author.

degree of correlation almost linearly declines with increased aggregation levels. Similarly, the corresponding t-values show a decline in the same direction. Secondly, it can be seen that the null hypothesis of zero correlation can be rejected, since the observed t-values are all substantially in excess of the critical t-values at $\alpha = 0.05$ level of significance.

In the second set of tests, logarithmic intermediate demand prediction errors at the 79 x 79 level of aggregation for the *industry technology* model were studied as a function of a number of *dimensional* variables that can be listed as follows: (a) primary product specialization ratio for each industry, (b) primary products of each industry as a proportion of total domestic production of that industry's primary products, wherever produced, and (c) total intermediate demand for domestically produced products as a proportion of total domestic supply of primary products. These *dimensional* variables are listed in tables given in Appendix F.

The interest on these *dimensional* variables came about as a result of Ghosh's experiments, reported in Chapter III, where he hypothesized a high degree of correlation between prediction errors and the extent to which an industry's products are shipped to others for intermediate consumption (i.e., the greater an industry's shipments to other industries for intermediate consumption as a proportion of its total domestic output, the smaller is the prediction error for that sector because of its high degree of *forward* linkage with other sectors).

The results of these tests were most disappointing, providing no clue to substantiate the type of hypothesis just mentioned. The r^2 in all cases were exceedingly low, the highest being 0.08590. A visual inspection for nonlinear relationships did not yield any results either, and the line of investigation was terminated.

TABLE IV-11

THE RELATIONSHIP BETWEEN THE ABSOLUTE VALUE OF THE INTERMEDIATE DEMAND PREDICTION ERRORS AND THE ACTUAL 1961 INTERMEDIATE DEMAND LEVELS, COMMODITY TECHNOLOGY MODEL

	Description	Aggregation Levels			
		I (79 x 79)	II (60 x 60)	III (45 x 45)	IV (17 x 17)
1.	Number of Observations Used	75	57	41	14
2.	Sectors for which Observations have been Omitted in the Analysis	24, 25, 26, 73	14, 15, 56	13, 14, 15, 41	4, 7, 15
3.	Least-Squares Regression Equation Y : Prediction Errors X : Actual 1961 Values	Y = 69.29683 + .0342060X	Y = 112.49539 + .0311096X	Y = 141.87078 + .0321279X	Y = 289.35455 + .0440629X
4.	r^2	.56420	.50138	.46948	.40978
5.	t-Ratio	9.7215502	7.4367021	5.8748039	2.8864229
6.	Critical Value of t-Ratio at 5% Level of Significance	1.67	1.67	1.69	1.77

TABLE IV-12

THE RELATIONSHIP BETWEEN THE ABSOLUTE VALUE OF THE INTERMEDIATE DEMAND PREDICTION ERRORS AND THE ACTUAL 1961 INTERMEDIATE DEMAND LEVELS, INDUSTRY TECHNOLOGY MODEL

		Aggregation Levels			
		I	II	III	IV
	Description	(79 x 79)	(60 x 60)	(45 x 45)	(17 x 17)
1.	Number of Observations Used	79	60	44	16
2.	Sectors for which Observations have been Omitted in the Analysis	none	none	15	7
3.	Least-Squares Regression Equation Y : Prediction Errors X : Actual 1961 Values	Y = 55.45160 + .0391075X	Y = 103.34618 + .0431992X	Y = 176.06932 + .0302307X	Y = 287.17206 + .0291532X
4.	r^2	.57120	.49169	.40482	.17384
5.	t-Ratio	10.1276710	7.4902176	5.3447610	1.7163566
6.	Critical Value of t-Ratio at 5% Level of Significance	1.67	1.67	1.69	1.75

F. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

The basic conclusions reached from the input-output experiments reported in this chapter can be listed as follows:

1. When the results are analyzed by making no particular set of assumptions on the structure of measurement errors inherent in the two models, the available evidence would seem to suggest that the *industry technology* model gives a generally better predictive performance than the *commodity technology* model, over all aggregation levels taken together, on the basis of a number of measures of overall predictive performance. No significant differences are observable, however, when overall predictive performance is measured in terms of *weighted mean logarithmic prediction error.*

A detailed investigation of the underlying causes of the somewhat inferior predictive performance of the *commodity technology* model, over all aggregations, shows that this result may have been due largely to a particular procedure applied in constructing the basic intersectoral flows information used in developing the model. Specifically, it would seem that the treatment of secondary products of certain industries, such as advertising services rendered by the printing and publishing industry, causes certain perturbations in the underlying matrices, which, in turn, lead to either large over-predictions or under-predictions for a few sectors. This consequently makes the *commodity technology* model look as though it does not perform as well as the *industry technology* model.

Thus, it is important to underline that the application of a somewhat different procedure in empirically constructing the intersectoral flows information for the *commodity technology* model could well have turned the results in its favor.

2. When the models are compared at each level of aggregation separately, it becomes difficult to judge one model to be *better* than the other, since the various measures of predictive performance show important differences. It would seem that if the researcher is interested primarily in the geometric mean of the prediction errors, he would be more or less indifferent as to which model he chooses at approximately the 50 x 50 and 55 x 55 levels of aggregation. On the other hand, and under the same criterion, he would choose the *commodity technology* model when the model contains roughly 55 or more sectors and would certainly choose the *industry technology* model if the model has less than about

45 sectors. If, on the other hand, the researcher is primarily interested in reducing the size of the absolute value of the prediction errors, he would be inclined to use the *industry technology* model at all levels of aggregation.

3. The conclusions reached above must be conditioned by the fact that the structure of measurement errors inherent in the models used in the experiments is unknown. The model parameters have been obtained from single samples, so that even if the estimates can be considered as best estimates, no information is available on the variances. Since it is highly unlikely, however, that useful information on the measurement errors inherent in these models will become available, sensitivity tests of the type that are essentially presented here should be of help in assessing the empirical properties of alternative input-output modeling approaches.

4. There appears to be a definite nonlinear relationship between the level of sectoral aggregation and the respective measures of overall predictive performance, such that in general, the prediction error rises rapidly as the model is made more aggregated. The definite, reverse J-type curve can be approximated by fitting perhaps a third degree polynomial.

5. There seems to be a correlation (by virtue of rejecting the null hypothesis of zero correlation) between the absolute value of intermediate demand prediction errors and the actual (to be predicted) intermediate demand levels, at all levels of aggregation. The r^2, as well as the corresponding t-value, both decline more or less linearly with increasing levels of aggregation.

On the other hand, logarithmic prediction errors do not seem to be significantly correlated with a number of *dimensional* variables, such as the proportion of total domestic output of a given product used for intermediate consumption, primary product specialization of industries, etc.

The conclusions reached above are based on experiments in which only the *exogenous* versions of the models developed early in this chapter have been used. It seems that further experiments, using the *endogenous* versions of the models, where competitive imports are treated endogenously, would be helpful in verifying some of the conclusions reached here. Also, alternative sectoral aggregation procedures can be used to see how the observations on the effects of aggregation would differ. In addition, it would be desirable to have more observation points to be on stronger grounds in testing for various hypotheses.

Other areas of research are open in input-output modeling. For example, the efficiency of alternative methods for updating the *technological* coefficients, such as Stone's RAS method and perhaps other methods, would be enlightening.

CHAPTER V

ALTERNATIVE MODEL DESIGNS FOR NATIONAL, REGIONAL, AND MULTIREGIONAL ECONOMIC ANALYSIS

A. INTRODUCTION

The *commodity* and *industry* technology models developed and used in the experiments reported in the last chapter reflect two basic *informational* assumptions that underlie their particular mathematical structure. First, the *technology* matrix in each of these models consists of a *combined* matrix, where the *domestic technology* matrix and the *direct competitive imports requirements* matrix are not observed separately but only as a single matrix. Secondly, the final demand vector consists only of final consumption demand for domestically produced products, while the final demand for competitive imports is ignored in the sense that the product composition of competitive imports delivered to the final demand sectors is not known. Both of these assumptions stem from the particular manner in which the basic information required in input-output model construction has been organized in the 1958 Input-Output Study of the United States. This suggests that under alternative conditions concerning the availability of the various types of information required in input-output model construction, alternative models, each having a different mathematical structure, can be developed for national, as well as regional economic analysis. The distinction between *regional* and *national* can be dropped, since conceptually both terms refer to the *one-region* or *single-region* case. The alternative models developed for the one-region case can be further extended into the two-region, three-region, etc., cases for multiregional analysis.

In this chapter, five alternative pairs of *commodity technology* and *industry technology* models are developed for the one-region case, under alternative sets of assumptions on the measurement of the *technology* matrix, the treatment of competitive and noncompetitive imports, and the availability of information required in model construction, particularly with respect to the measurement of the final demand vector. The discussion on these models for the one-region case is followed by a brief review of some of the more well-known inter-regional models that have been developed in the past. Finally, the *commodity technology* model developed earlier under Alternative V is extended into the two-region case, to illustrate through a concrete example how the five sets of one-regional models can be further extended for multiregional analysis.

A review of the input-output literature shows that the type of systematic explorations on alternative model designs as presented here is completely lacking at the present. Perhaps the only work that can be cited in this respect is that by Matuszewski, Pitts, and Sawyer,[1] which serves as a point of departure for the model formulations given in this chapter. Generally, the regional input-output literature, despite its considerable volume has been content with "the state of the arts" in model formulation, and not much has been done during the last two decades to change the situation significantly. A detailed review of the regional input-output literature at this point would not be warranted. However, a very brief *tour* of the literature will orient the reader to this field and help him understand the general context of the models developed in this chapter.

Among input-output models constructed at the regional level, the two-region study conducted by Chenery, Clark and Cao-Pinna for Italy,[2] Artle's study of the structure of the Stockholm economy,[3] the Moore and Petersen study of Utah,[4] and the St. Louis study by

[1] T.I. Matuszewski, Paul R. Pitts, and John A. Sawyer, "Alternative Treatments of Imports in Input-Output Models: A Canadian Study," *Journal of the Royal Statistical Society*, Series A, CXXVI (1963), 410-432; and "Inter-Industry Estimates of Canadian Imports, 1949-1958," in W. C. Hood and John A. Sawyer (eds.), *Canadian Political Science Association Conference on Statistics, 1961*, held at Sir George Williams University, Montreal, Canada (Toronto: University of Toronto Press, 1963), pp. 140-167.

[2] Hollis B. Chenery, Paul G. Clark, and Vera Cao-Pinna, *The Structure and Growth of the Italian Economy* (Rome: U.S. Mutual Security Agency, 1953).

[3] Roland Artle, *The Structure of the Stockholm Economy* (Ithica, N.Y.: Cornell University Press, 1965).

[4] Frederick T. Moore and James W. Petersen, "Regional Analysis: An Interindustry Model of Utah," *The Review of Economics and Statistics,* XXXVII, 4(November, 1955), 358-383.

Hirsch[5] stand out as *classical* examples. More recent examples are provided by numerous input-output studies for states, regions, and urban areas.[6]

Some of the current work in regional input-output analysis includes the Boston Metropolitan Area input-output study by the Regional Science Research Institute, the Providence, Rhode Island, study by Brown University, Department of Economics, the Atlantic Provinces (Canada) study by Prof. Kari Levitt of McGill University and others, and the Quebec (Canada) study by Prof. T. I. Matuszewski of Laval University and his associates.

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[5]Werner Z. Hirsch, "The Interindustry Relations of a Metropolitan Area," *The Review of Economics and Statistics,* XLI, 4 (November, 1959), 360-369. Also see his "An Application of Area Input-Output Analysis," *The Regional Science Association: Papers and Proceedings, V (1959), 79-92;* and "Application of Input-Output Techniques to Urban Areas," in Tibor Barna (ed.), *Structural Interdependence and Economic Development,* Proceedings of an International Conference on Input-Output Techniques, Geneva, September 1961 (London: MacMillan and Co., Ltd., and New York: St. Martin's Press, 1963), 151-168.

[6]See, for example, the following: Robert L. Allen and Donald A. Watson, *The Structure of the Oregon Economy: An Input-Output Study* (Eugene, Oregon: University of Oregon, Bureau of Business and Economic Research, 1965);

J.G.D. Carden, "Input-Output Analysis for Mississippi," *Mississippi's Business,* XX, 4 (February 1964), 1-8;

W. Lee Hansen and Charles W. Tiebout, "An Intersectoral Flows Analysis of the California Economy," *The Review of Economics and Statistics,* XLV (November, 1963), 409-418;

Floyd K. Harmston and Richard E. Lund, *An Application of an Input-Output Framework to a Community Economic System* (Columbia, Missouri: University of Missouri Press, 1967);

Walter Isard, Thomas W. Langford, Jr., and Eliahu Romanoff, *Philadelphia Region Input-Output Study,* Preliminary Working Papers, Two Volumes (Philadelphia: University of Pennsylvania, Wharton School, Department of Regional Science, 1966).

C.D. Kirksey, *An Interindustry Study of the Sabine-Neches Area of Texas* (Austin, Texas: University of Texas, Bureau of Business Research, 1959);

T.H. Lee, John R. Moore, and David P. Lewis, *A Report on the Tennessee Interindustry Study* (Knoxville, Tennessee: The University of Tennessee, College of Business Administration, Center for Business and Economic Research, December, 1967);

William E. Martin and Harold O. Carter, *A California Interindustry Analysis Emphasizing Agriculture,* Giannini Foundation of Agricultural Economics, Research Report No. 250 (February 1962), Berkeley, California;

William H. Miernyk, Ernest Bonner, John H. Chapman, Jr., and Kenneth Shellhammer, *The Impact of Space and Space-Related Activities on a Local Community: Part I, The Input-Output Analysis,* a report submitted to the National Aeronautics and Space Administration (July, 1965);

Sang O. Park, "The Input-Output Method of Analyzing the State's Economy [New Mexico], *New Mexico Business,* XVI (September, 1963).

Amanda S. Rao and David J. Allee, *An Application of Interindustry Analysis to San Benito County* [California], Giannini Foundation of Agricultural Economics, Research Report No. 278 (September, 1964).

Excellent discussions of some of the major empirical problems faced in regional input-output analysis have been given by Cao-Pinna[7] and Levitt.[8]

The application of the input-output approach in regional economic forecasting has been illustrated by Berman, *et al.*,[9] Hoch,[10] and Lee.[11] Among the regional economic impact studies using input-output models, some of the more well-known are those by Isard and Kuenne,[12] Isard and Schooler,[13] Miller,[14] and Peterson and Tiebout.[15]

Within the context of a sample of the regional input-output literature just cited, the alternative model designs developed in this chapter should not only bring the conceptual and empirical clarity that is currently lacking in input-output analysis but also provide far greater flexibility and range of choice in regional and multiregional input-output model construction and applications.

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[7]Vera Cao-Pinna, "Problems of Establishing and Using Regional Input-Output Accounting," in Walter Isard and John H. Cumberland (eds.), *Regional Economic Planning, Techniques of Analysis for Less Developed Areas* (Paris: The European Productivity Agency of the Organisation for European Economic Co-Operation, July, 1961), pp. 305-324.

[8]Kari Levitt, "Inter-Industry Study of the Economy of the Atlantic Provinces," in Canadian Political Science Association *Papers on Regional Statistical Studies,* Conference on Statistics, 1964, held at Prince of Wales College, Charlottetown, P.E.I. (Toronto: University of Toronto Press, 1966), 151-198;

[9]B.R. Berman, B. Chinitz and E.M. Hoover, *Projection of a Metropolis,* Technical Supplement to the New York Metropolitan Region Study (Cambridge, Mass.: Harvard University Press, 1960).

[10]Irving Hoch, "A Comparison of Alternative Inter-Industry Forecasts for the Chicago Region," *The Regional Science Association: Papers and Proceedings, V (1959), 217-235.*

[11]Ivan M. Lee, *Conditional Projections of California Economic Growth,* Giannini Foundation Monograph Number 19 (Berkeley, California: University of California Division of Agricultural Sciences and Giannini Foundation of Agricultural Economics, February, 1967);

[12]Walter Isard and Robert E. Kuenne, "The Impact of Steel Upon the Greater New York-Philadelphia Industrial Region: A Study in Agglomeration Projection," *The Review of Economics and Statistics,* XXXV (November, 1953), 289-301.

[13]Walter Isard and Eugene W. Schooler, "An Economic Analysis of Local and Regional Impacts of Reduction of Military Expenditures," Paper presented at the Peace Research Conference, Kellogg Center, University of Chicago, November 18-19, 1963.

[14]Ronald E. Miller, "The Impact of the Aluminum Industry on the Pacific Northwest: A Regional Input-Output Analysis," *The Review of Economics and Statistics,* XXXIX (May, 1957), 200-209.

[15]Richard S. Peterson and Charles M. Tiebout, "Measuring the Impact of Regional Defense-Space Expenditures," *The Review of Economics and Statistics,* XLVI, 4 (November, 1964), 421-428.

B. ALTERNATIVE MODEL DESIGNS FOR NATIONAL OR REGIONAL (ONE-REGION) ECONOMIC ANALYSIS

The alternative model designs explored here represent extensions of the models developed and used in the experiments reported in the last chapter. As in the last chapter, the distinction between *commodity technology* and *industry technology* models is retained. To provide a reasonable limit to the number of extensions, competitive imports are assumed to be treated endogenously in the system for analytical or prediction purposes. Further, noncompetitive imports are assumed to be treated exogenously. However. in the last of the five alternative extensions explored here, noncompetitive imports are assumed to be treated endogenously, to illustrate how such an assumption leads to still more model designs and results in different mathematical formulations.

The five alternative sets of conditions under which the models are developed represent combinations of the following specific conditions or assumptions:

(a) The regional *technology* matrix A^D and the *competitive* imports direct input requirements matrix M are observable separately,

(b) A^D and M are observable only as a single, combined matrix,

(c) The final demand vector for regionally produced products Y^D and the final demand vector for competitive imports Y^M are observable as two separate vectors.

(d) The vectors Y^D and Y^M are observable only as a combined, single vector, and

(e) Noncompetitive imports required for intermediate and final consumption are treated endogenously, rather than exogenously.

The particular combinations of these conditions or assumptions leading to the five model design alternatives can be summarized as on the following page.

	Y^D and Y^M are observable separately	Y^D and Y^M are observable only as a single vector
A^D and M are observable as two separate matrices	Alternative I	Alternative II
A^D and M are observable only as a single, combined matrix	Alternative III	Alternative IV
Noncompetitive imports are treated endogenously; A^D and M are observable only as a single, combined matrix	Alternative V	This alternative is not explored

It can be seen that the number of alternative sets of conditions that lead to different model formulations increases rapidly with the addition of other assumptions on the availability of the types of information required in input-output model construction. Each of the five alternatives noted here is explored, first, in terms of a *commodity technology* model and, secondly, in terms of an *industry technology* model.

These models are *recursive*, in the sense that the vector of regional production levels are first numerically determined and then used in solving for the competitive import requirements vector.

Models developed under Alternatives III, IV, and V allow for perfect substitutability between regionally produced products and competitive imports for *intermediate* consumption, whereas under Alternatives I and II no such substitution is permitted.

The flexibility provided by the perfect substitutability assumption under Alternatives III, IV, and V is rendered slightly limited by another, simultaneous assumption that the respective market shares of the regionally produced products and competitive imports used for intermediate consumption are fixed at the base year values.

Models developed under conditions II and IV cannot be used to analyze the effects of a shift in final demand from regionally produced products to competitive imports or *vice versa* (i.e., import substitution) *on* regional production levels and competitive imports requirements. Models developed under Alternatives I, III, and V are well suited for this type of sensitivity or impact analysis.

1. Alternative I

We can extend both the *commodity technology* model (Model I) and the *industry technology* model (Model II) developed in the last chapter by assuming, *first*, that the *regional technology* matrix A^D and the *competitive imports direct input requirements* matrix M are observable as two separate matrices, and *secondly*, by assuming that the final demand vector for *regionally* produced products Y^D and the vector of competitive imports required for final consumption Y^M are observable (and predictable) as two separate vectors. Noncompetitive imports are assumed to be treated exogenously.

a. Alternative I: "Commodity Technology" Model

Referring to Eq.(4.5) in Chapter IV, we can write the fundamental commodity flow balance equations for the *commodity technology* model as follows:

$$(5.1) \qquad \begin{aligned} x_i^D - \sum_{j=1}^{n} a_{ij}^D x_j^D - \sum_{k=1}^{n} h_{ik} x_k^M &= y_i^D \\ - \sum_{j=1}^{n} m_{ij} x_j^D + \qquad\qquad x_i^M &= y_i^M \end{aligned}$$

where the superscript D means *regional* and the superscript M denotes *competitive imports*, and where x_i^D is the total *regional* output of product or commodity i ($x_i^D = x_j^D$ for i=j); $a_{ij}^D = x_{ij}^D/x_j^D$ is the *regional* input-coefficient; $m_{ij} = x_{ij}^M/x_j^D$ is the competitive imports input-coefficient (i.e., competitive imports direct input requirements coefficient); y_i^M is total competitive imports of product i required for final consumption; $x_i^M = \sum_{j=1}^{n} x_{ij}^M + y_i^M$ is total competitive imports requirements for product i ($x_i^M = x_k^M$ for i = k); h_{ik} is a parameter that represents the total additional or extra *regional* output of type i (i.e., transportation and warehousing, wholesale and retail trade, finance and insurance) required per unit of competitive imports of each product (k = 1, 2, ..., n) used for intermediate *and* final consumption. Defined in this manner, the h_{ik} coefficients are slightly different from the way they were defined in Eq. (4.6) in Chapter IV. The denominator in Eq. (4.6), which was $\underset{\sim}{x}_i^M$, is here replaced by x_i^M. It should be noted that $\underset{\sim}{x}_i^M = \sum_{j=1}^{n} x_{ij}^M$, while $x_i^M = \sum_{j=1}^{n} x_{ij}^M + y_i^M$. Furthermore, the numerator is defined differently, to include not only $\sum_{j=1}^{n} {}_r\Delta x_{ij}^{D(M)}$ (i.e., total additional *regional* output of *margin* service of type r required to deliver competitive imports of type i to *regional* sectors j = 1,...,n which use the competitive imports as intermediate goods), but also ${}_r\Delta y_i^{D(M)}$ (i.e., total additional *regional* output of *margin* service of type r required to deliver competitive imports of type i to the *regional* final demand sectors).

In summary, the derivation of the h_{ik} coefficients involves a two step process. *First*, we define

$$ {}_r h_i = \frac{\sum_{j=1}^{n} {}_r\Delta x_{ij}^{D(M)} + {}_r\Delta y_i^{D(M)}}{x_i^M} $$

where each ${}_r h_i$ represents the additional *regional* output of *margin* services of type r required to deliver competitive imports of type i to both intermediate and final consumers. It can be seen that for each *margin* service of type r (e.g., transportation), we have a column of h_i ($i = 1,2,\ldots,n$) coefficients, or r columns in total. *Secondly*, we can write each such column as a row, using the subscript i for rows and k for columns. We can thus define the nxn matrix $H = [h_{ik}]$, in which all rows except for the r *margin* rows are zero and in which the columns corresponding to the *margin* rows contain zero elements.

Writing Eq. (5.1) more compactly, we have

$$ (5.2) \qquad \begin{bmatrix} (I - A^D) & -H \\ -M & I \end{bmatrix} \begin{pmatrix} X^D \\ X^M \end{pmatrix} = \begin{pmatrix} Y^D \\ Y^M \end{pmatrix} $$

where the matrices A^D, H, and M correspond to the lower case parameters in Eq. (5.1), I is the nxn identity matrix, and the vectors X^D, X^M, Y^D, and Y^M correspond to the lower case variables in Eq. (5.1).[16]

[16] The reader is advised to refer to the explanations accompanying Eqs. (4.6) and (4.7) in Chapter IV for a review of the H matrix and its construction.

Finally, Eq. (5.2) can be expressed in *reduced form* as

$$\text{(5.3)} \qquad \begin{pmatrix} X^D \\ \\ X^M \end{pmatrix} = \begin{bmatrix} (I - A^D) & -H \\ \\ -M & I \end{bmatrix}^{-1} \begin{pmatrix} Y^D \\ \\ Y^M \end{pmatrix}$$

which, after matrix inversion [refer to Appendix D] results in

$$\text{(5.4)} \qquad \begin{pmatrix} X^D \\ \\ X^M \end{pmatrix} = \begin{bmatrix} [(I - A^D) - HM]^{-1} & [(I - A^D) - HM]^{-1}H \\ \\ M\,[(I - A^D) - HM]^{-1} & I + M\,[(I - A^D) - HM]^{-1}H \end{bmatrix} \begin{pmatrix} Y^D \\ \\ Y^M \end{pmatrix}$$

The *upper-left submatrix* in Eq. (5.4) helps us determine total regionally production requirements for each commodity, to satisfy not only the final demand for *regionally* produced products but also the total (direct and indirect) intermediate demand for goods and services generated by that final demand. The *upper-right submatrix* specifies the total (direct and indirect) contribution made by a unit vector of competitive imports used for final consumption to the *regional* economy, through competitive imports-induced *regional* distributive services. When we assume that an increase in final demand for competitive imports represents a shift in final demand from *regionally* produced products to competitive imports, the *upper-right submatrix* shows how much of a reduction in the *regional* production of each product will result per unit of such shift for each product. The *lower-left submatrix* specifies the total (direct and indirect) competitive imports requirements of the economy attributable to a unit vector of final demand for *regionally* produced products. Finally, the *lower-right submatrix* shows the total (direct and indirect) competitive imports requirements of the *regional* economy attributable to a unit vector of competitive imports for final consumption.

When we assume, again, that a rise in Y^M represents a shift from Y^D to Y^M, the *lower-right submatrix* specifies the decrease that will result in X^M (i.e., total competitive imports requirements vector) due to a shift from Y^M to Y^D (i.e., due to import substitution for products entering the system for final consumption). Clearly, the *matrix multipliers* represented by these four submatrices should be of considerable importance in regional or national economic analysis, since they quantitatively specify in great detail the direct and indirect production relationships that tie together *regional* production, consumption, and external trade.

The chief drawback of the model just outlined is that no substitution is assumed to exist between *regionally* produced products and competitive imports for either intermediate or final consumption. This drawback, however, can be overcome quite easily by formulating the model in other ways, as will be shown. The main advantage offered by the model is that the competitive imports input matrix M is fully spelled out.

b. Alternative I: "Industry Technology" Model

Under the same assumptions as before, we can write the commodity flow balance equations for the *industry technology* model as follows:

$$
\begin{aligned}
x_i^D - \sum_{j=1}^{n} a_{ij}^D \tilde{x}_j^D - \sum_{k=1}^{n} h_{ik} x_k^M &= y_i^D \\
- \sum_{j=1}^{n} m_{ij} \tilde{x}_i^D + \qquad\qquad x_i^M &= y_i^M
\end{aligned}
\tag{5.5}
$$

which is exactly the same as Eq. (5.1) except for the introduction here of $\tilde{x}_j^D$ (i.e., vector of *regional industry* output levels) and the changes in the definition of the a_{ij}^D and m_{ij} coefficients that it inevitably causes. We now have a_{ij}^D defined as $a_{ij}^D = x_{ij}^D/\tilde{x}_j^D$ and m_{ij} defined as $m_{ij} = x_{ij}^M/\tilde{x}_j^D$. Writing Eq. (5.5) in more compact notations, we now have

$$
\begin{pmatrix} X^D \\ \\ X^M \end{pmatrix} = \begin{bmatrix} A^D & H \\ \\ M & 0 \end{bmatrix} \begin{pmatrix} \tilde{X}^D \\ \\ X^M \end{pmatrix} + \begin{pmatrix} Y^D \\ \\ Y^M \end{pmatrix}
\tag{5.6}
$$

It is necessary to introduce, at this point, the *industry product mix* matrix $U = [u_{k\ell}]$, in which each element $u_{k\ell}$ represents the proportion of the total domestic production of commodity ℓ that is produced by *industry* k [refer to Chapter IV, Section B, Part 2]. After replacing the vector $\widetilde{X}^D$ by UX^D [refer to Eq. (4.16) in Chapter IV], and following the same type of mathematical manipulations as shown in detail in Eqs. (4.17) through (4.22) in Chapter IV, we end up with

$$(5.7)\qquad \begin{pmatrix} X^D \\ \\ X^M \end{pmatrix} = \begin{bmatrix} (I - A^D U) & -H \\ \\ -MU & I \end{bmatrix}^{-1} \begin{pmatrix} Y^D \\ \\ Y^M \end{pmatrix}$$

which, upon matrix inversion [refer to Appendix D], becomes

$$(5.8)\qquad \begin{pmatrix} X^D \\ \\ X^M \end{pmatrix} = \begin{bmatrix} [(I - A^D U) - HMU]^{-1} & [(I - A^D U) - HMU]^{-1} \\ \\ MU[(I - A^D U) - HMU]^{-1} & I + MU[(I - A^D U) - HMU]^{-1}\, H \end{bmatrix} \begin{pmatrix} Y^D \\ \\ Y^M \end{pmatrix}$$

which has the same economic interpretation as Eq. (5.4).

2. Alternative II

A second extension of both the *commodity technology* model (Model I) and the *industry technology* model (Model II) is possible when we assume, *first*, that the *domestic technology* matrix A^D and the *competitive imports direct requirements* matrix M are observable as two separate matrices, and *secondly,* that the final demand vector for regionally produced products Y^D and the vector of competitive imports required for final consumption Y^M are observable (and predictable) *not* as two separate vectors (as previously assumed) but only as a *combined* vector (i.e., $Y = Y^D + Y^M$).

a. Alternative II: "Commodity Technology" Model

Under these circumstances, the commodity flow balance equations in (5.1) now become

$$(5.9)\qquad x_i^D - \sum_{j=1}^{n} a_{ij}^D\, x_j^D - \sum_{k=1}^{n} h_{ik}\, x_k^M + (x_i^M - \sum_{j=1}^{n} x_{ij}^M) = (y_i^D + y_i^M)$$

$$\sum_{j=1}^{n} m_{ij}\, x_j^D \qquad - \qquad k_i\, x_i^M = 0$$

where

$x_i^D = x_j^D$ for $i = j$, $i, j = 1,2,\ldots,n$;

$x_i^M = x_k^M$ for $i = k$, $k = 1,2,\ldots,n$;

$(x_i^M - \sum_{j=1}^{n} x_{ij}^M) = y_i^M$,

$k_i = \sum_{j=1}^{n} x_{ij}^M / x_i^M$,

and where the other terms are as explained earlier.

Replacing $\sum_{j=1}^{n} x_{ij}^M$ in (5.9) by $\sum_{j=1}^{n} m_{ij}\, x_j^D$, consolidating terms, and writing it more compactly, we have

$$(5.10)\qquad \begin{bmatrix} (I - A^D - M) & (I - H) \\ M & -\hat{k} \end{bmatrix} \begin{pmatrix} X^D \\ X^M \end{pmatrix} = \begin{pmatrix} (Y^D + Y^M) \\ 0 \end{pmatrix}$$

where

$\hat{k} = [k_i]$ is a diagonal matrix and where, as before, each submatrix is of the order nxn. After matrix inversion, Eq. (5.10) becomes

$$(5.11)\quad \begin{pmatrix} X^D \\ X^M \end{pmatrix} = \begin{bmatrix} [(I-A-M)+(I-H)\hat{k}^{-1}M]^{-1} & [(I-A-M)+(I-H)\hat{k}^{-1}M]^{-1}\,(I-H)\hat{k}^{-1} \\ \hat{k}^{-1}M[(I-A-M)+(I-H)\hat{k}^{-1}M]^{-1} & -\hat{k}^{-1}+\hat{k}^{-1}M\,[(I-A-M)+(I-H)\hat{k}^{-1}M]^{-1}\,(I-H)\hat{k}^{-1} \end{bmatrix} \begin{pmatrix} (Y^D+Y^M) \\ O \end{pmatrix}$$

where the four submatrices have the same economic interpretation as in Eq. (5.4).

As in Extension I, it is assumed that no substitution exists between *regionally* produced products and competitive imports used for intermediate consumption. Since the final demand vector is measured as a *combined* vector (i.e., $(Y^D + Y^M)$), substitution is implicitly assumed between *regionally* and externally produced products for final consumption. Because of the latter, perfect substitution assumption, the model does not distinguish between the economic effects of shifts in final demand for domestically produced products or competitive imports. To this extent, the analytical usefulness of the model is somewhat limited.

b. Alternative II: "Industry Technology" Model

Under the same assumptions as before (i.e., the two matrices A^D and M are observed separately while the final demand vector $Y = Y^D + Y^M$ is observed only as a combined vector), the *industry technology* model can be developed by first writing the commodity balance equations as follows:

$$(5.12)\quad \begin{array}{l} x_i^D - \sum_{j=1}^{n} a_{ij}^D \tilde{x}_j^D - \sum_{k=1}^{n} h_{ik}\, x_k^M + (x_i^M - \sum_{j=1}^{n} x_{ij}^M) = (y_i^D + y_i^M) \\ x_i^M + \sum_{j=1}^{n} m_{ij}\, \tilde{x}_j^D - x_i^M \qquad\qquad - k_i\, x_i^M \;=\; 0 \end{array}$$

where

$$x_i^M = x_k^M \quad \text{for} \quad i = k$$

$$(x_i^M - \sum_{j=1}^{n} x_{ij}^M) = y_i^M$$

$$k_i = \sum_{j=1}^{n} x_{ij}^M / x_i^M$$

and where x_i^D refers to total *regional* output of product i whereas $\tilde{x}_j^D$ refers to the total *regional* output of *industry* j, so that $x_i^D \neq \tilde{x}_j^D$ for i = j. The other terms are as explained earlier. Replacing $\sum_{j=1}^{n} x_{ij}^M$ in (5.12) by $\sum_{j=1}^{n} m_{ij}\, \tilde{x}_j^D$, consolidating terms, and writing it more compactly, we have

$$(5.13)\quad \begin{pmatrix} X^D \\ \\ X^M \end{pmatrix} = \begin{bmatrix} (A^D + M) & (H - I) \\ \\ -M & (I + \hat{k}) \end{bmatrix} \begin{pmatrix} \tilde{X}^D \\ \\ X^M \end{pmatrix} + \begin{pmatrix} (Y^D + Y^M) \\ \\ O \end{pmatrix}$$

Introducing, again, the *industry product mix* matrix $U = [u_{k\ell}]$, such that $\tilde{X}^D = UX^D$, and replacing the vector $\tilde{X}^D$ by UX^D, we can re-write Eq. (5.13) as

$$(5.14)\quad \begin{pmatrix} X^D \\ \\ X^M \end{pmatrix} = \begin{bmatrix} (A^D + M)\, U & (H - I) \\ \\ -MU & (I + \hat{k}) \end{bmatrix} \begin{pmatrix} X^D \\ \\ X^M \end{pmatrix} + \begin{pmatrix} (Y^D + Y^M) \\ \\ O \end{pmatrix}$$

Following the same type of mathematical manipulations as described in Eqs. (4.17) through (4.22) in Chapter IV, we can re-write Eq. (5.14) as

$$(5.15)\quad \begin{bmatrix} [I - (A^D + M)\, U] & (I - H) \\ \\ MU & -\hat{k} \end{bmatrix} \begin{pmatrix} X^D \\ \\ X^M \end{pmatrix} = \begin{pmatrix} (Y^D + Y^M) \\ \\ O \end{pmatrix}$$

which, after matrix inversion, becomes

$$(5.16)\quad \begin{pmatrix} X^D \\ \\ X^M \end{pmatrix} = \begin{bmatrix} \left\{[I-(A^D+M)U]+(I-H)\hat{k}^{-1}MU\right\}^{-1} & \left\{[I-(A^D+M)U]+(I-H)\hat{k}^{-1}MU\right\}^{-1}(I-H)\hat{k}^{-1} \\ \\ \hat{k}^{-1}MU\left\{[I-(A^D+M)U]+(I-H)\hat{k}^{-1}MU\right\}^{-1} & -\hat{k}^{-1}+\hat{k}^{-1}MU\left\{[I-(A^D+M)U]+(I-H)\hat{k}^{-1}MU\right\}^{-1}(I-H)\hat{k}^{-1} \end{bmatrix} \begin{pmatrix} (Y^D+Y^M) \\ \\ O \end{pmatrix}$$

where the four submatrices have the same economic interpretation as in Eq. (5.4). The *upper-right* and *lower right* submatrices fall out since they are post-multiplied by a null vector. The *upper-left submatrix* transforms the *combined* bill of goods for *regionally* produced products and competitive imports *into* total (i.e., direct and indirect) *regional* output requirements. The *lower-left submatrix* helps us determine the direct and indirect requirements of the economy for competitive imports.

The *industry technology* version of Alternative II is subject to the same type of limitation as already noted in connection with the *commodity technology* version. Namely, the analytical usefulness of the model is somewhat limited by the assumption of perfect substitutability between regionally produced products and competitive imports used for final consumption.

3. Alternative III

A third extension of both the *commodity technology* model and the *industry technology* model can be developed by assuming, *first*, that the *technology* matrix is observable only as a *combined* matrix, rather than as two separate matrices, and *secondly*, that the final demand vector for regionally produced products Y^D and the vector of competitive imports required for final consumption Y^M are observable (predictable) as two separate vectors. These assumptions differ from those underlying the models used in the experiments, since in the latter, the vector of competitive imports required for final consumption was actually not available as a result of the empirical procedures followed in the 1958 Input-Output Study of the United States.

a. Alternative III: "Commodity Technology" Model

Under the assumptions just noted, the familiar commodity flow balance equations of the *commodity technology* model can be written as follows:

$$(5.17)\qquad \begin{aligned} x_i^D - \sum_{j=1}^{n} (a_{ij}^D + m_{ij})\, x_j^D + g_i\, x_i^D - \sum_{k=1} h_{ik}\, x_k^M &= y_i^D \\ - g_i\, x_i^D \qquad\qquad\qquad\qquad + x_i^M &= y_i^M \end{aligned}$$

where

$$g_i = \frac{\sum_{j=1}^{n} x_{ij}^M}{x_i^D}.$$

and where the combined matrix $(a_{ij}^D + m_{ij})$ is as defined earlier. Writing Eq. (5.17) more compactly, we have

$$(5.18)\qquad \begin{bmatrix} [I - (A^D + M) + \hat{g}] & -H \\ -\hat{g} & I \end{bmatrix} \begin{pmatrix} X^D \\ X^M \end{pmatrix} = \begin{pmatrix} Y^D \\ Y^M \end{pmatrix}$$

where, as before, $\hat{g}$ is a diagonal matrix consisting of g_i.

After matrix inversion, Eq. (5.18) becomes

$$(5.19)\qquad \begin{pmatrix} X^D \\ X^M \end{pmatrix} = \begin{bmatrix} \left\{[I - (A^D + M) + \hat{g}] - H\hat{g}\right\}^{-1} & \left\{[I - (A^D + M) + \hat{g}] - H\hat{g}\right\}^{-1} H \\ \hat{g}\left\{[I - (A^D + M) + \hat{g}] - H\hat{g}\right\}^{-1} & I + \hat{g}\left\{[I - (A^D + M) + \hat{g}] - H\hat{g}\right\}^{-1} H \end{bmatrix} \begin{pmatrix} Y^D \\ Y^M \end{pmatrix},$$

where the four submatrices have the same economic interpretation as in Eq. (5.4).

It can be seen that the results given in Eq. (5.19) are exactly the same as those given in Eq. (4.10) in Chapter IV. It should be noted, at the same time, that the two models have different information requirements and the predictions they yield are different.

b. Alternative III: "Industry Technology" Model

The *industry technology* model version of Extension 3 can be developed by writing the commodity flow balance equations (5.17) as (5.20)

$$(5.20) \qquad \begin{aligned} x_i^D - \sum_{j=1}^{n} (a_{ij}^D + m_{ij})\, \tilde{x}_j^D + g_i\, x_i^D - \sum_{k=1}^{n} h_{ik}\, x_k^M &= y_i^D \\ - g_i\, \tilde{x}_i^D \qquad\qquad\qquad\qquad + x_i^M &= y_i^M \end{aligned}$$

where

$$a_{ij}^D = \frac{x_{ij}^D}{\tilde{x}_j^D} \; ; \; m_{ij} = \frac{x_{ij}^M}{\tilde{x}_j^D} \; ; \; g_i = \frac{\sum_{j=1}^{n} x_{ij}^M}{\tilde{x}_i^D} \; ;$$

$$x_i^D \neq \tilde{x}_j^D \text{ for } i = j; \; x_i^M = x_k^M \text{ for } i = k;$$

$$\tilde{x}_i^D = \tilde{x}_j^D \text{ for } i = j.$$

To avoid duplication, it should be pointed out that the *industry technology* model that we can proceed to develop here will turn out to be exactly the same as the *industry technology* model developed and used in the experiments (refer to Chapter IV, Eqs. (4.13) through (4.24d)),when we substitute $X^M = (x_i^M)$ for $\underset{\sim}{X}^M = (\underset{\sim}{x}_i^M)$ and replace the null vector of competitive imports used for final consumption by $Y^M = (y_i^M)$.

The advantage offered by both the *commodity technology* and the *industry technology* versions of Extension III is that, when the *technology* matrix is observable only as a combined matrix, it becomes possible to trace the direct and indirect effects of competitive imports used for final consumption on *regional* production, as well as on the *regional* direct and indirect competitive imports requirements for intermediate consumption.

4. Alternative IV

A fourth extension of both the *commodity technology* model and the *industry technology* model is possible under the following two assumptions: *first,* the *technology* matrix is observable only as a *combined* matrix (i.e., the matrices A^D and M are collapsed into one matrix, so that they are not observable separately), and *second,* final demand for both domestically produced products and for competitive imports is observable only as a single, *combined* vector (i.e., $(Y^D + Y^M)$) rather than as two separate vectors. Under these assumptions, perfect substitution is allowed between domestically produced products and competitive imports, not only for intermediate (i.e., interindustry) but also for final consumption.

a. Alternative IV: "Commodity Technology" Model

These assumptions lead to the formulation of the commodity flow balance equations as follows:

$$(5.21)\quad \begin{array}{llll} x_i^D - \sum_{j=1}^{n} (a_{ij}^D + m_{ij})\, x_j^D + g_i\, x_i^D & - \sum_{k=1}^{n} h_{ik}\, x_k^M + (x_i^M - \sum_{j=1}^{n} x_{ij}^M) & = & (y_i^D + y_i^M) \\ g_i\, x_i^D & - k_i\, x_i^M & = & 0 \end{array}$$

where

$$k_i = \sum_{j=1}^{n} x_{ij}^M / x_i^M \; ; \; \sum_{j=1}^{n} x_{ij}^M = k_i\, x_i^M \; ;$$

$$g_i = \frac{\sum_{j=1}^{n} m_{ij}\, x_j^D}{x_i^D} = \frac{\sum_{j=1}^{n} x_{ij}^M}{x_i^D} = \frac{x_i^M}{x_i^D} \; ;$$

$k_i\, x_i^M = g_i\, x_i^D$, but obviously $k_i \neq g_i$.

In matrix notations, Eq. (5.21) can be re-written as

$$(5.22)\quad \begin{array}{llll} X^D - (A^D + M)\, X^D + \hat{g}\, X^D & + X^M - HX^M - \hat{k}X^M & = & (Y^D + Y^M) \\ \hat{g}\, X^D & - \hat{k}\, X^M & = & 0 \end{array}$$

where $\hat{g}$ and $\hat{k}$ are diagonal matrices consisting, respectively, of g_i and k_i. It should be noted in Eq. (5.22) that $X^M - \hat{k}X^M = Y^M$. Re-writing Eq.(5.22) we have

$$(5.23)\quad \begin{bmatrix} [\,I-(A^D+M)+\hat{g}\,] & (I-H-\hat{k}) \\ \hat{g} & -\hat{k} \end{bmatrix} \begin{pmatrix} X^D \\ X^M \end{pmatrix} = \begin{pmatrix} (Y^D+Y^M) \\ 0 \end{pmatrix}$$

which, after matrix inversion, results in

$$(5.24)\quad \begin{pmatrix} X^D \\ X^M \end{pmatrix} = \begin{bmatrix} \{[I-(A^D+M)+\hat{g}]+(I-H-\hat{k})\,\hat{k}^{-1}\,\hat{g}\}^{-1} & \{[I-(A^D+M)+\hat{g}]+(I-H-\hat{k})\,\hat{k}^{-1}\,\hat{g}\}^{-1}\,(I-H-\hat{k})\,\hat{k}^{-1} \\ \hat{k}^{-1}\hat{g}\,\{[I-(A^D+M)+\hat{g}]+(I-H-\hat{k})\,\hat{k}^{-1}\,\hat{g}\}^{-1} & -\hat{k}^{-1}+\hat{k}^{-1}\,\hat{g}\,\{[I-(A^D+M)+\hat{g}]+(I-H-\hat{k})\,\hat{k}^{-1}\,\hat{g}\}^{-1}\,(I-H-\hat{k})\,\hat{k}^{-1} \end{bmatrix} \begin{pmatrix} (Y^D+Y^M) \\ 0 \end{pmatrix}$$

The four submatrices here have the same economic interpretation as before, but they obviously reflect a different set of assumptions on the relationship between domestically produced products and competitive imports, as well as on the availability or *observability* of the various types of information required by the model. When the model is used to analyze the direct and indirect effects of certain hypothetical or actual shifts in final demand *on* the *regional* output levels and on the competitive imports requirements of the economy under consideration, the basic assumptions underlying the model as noted here must be clearly understood. Because of the perfect substitutability assumption concerning domestically produced products and competitive imports used for final consumption, the model cannot be used to trace back to the responsible sectors the net direct and indirect economic effects of, for example, an hypothetical import-substitution program. In short, the model is *source-of-supply-blind* in tracing out economic effects, and for this reason, it is not very helpful as an analytical tool for studying certain types of problems, particularly those related to foreign trade.

b. Alternative IV: "Industry Technology" Model

The commodity flow balance equations of the *industry technology* model can be written as

$$(5.25)\quad \begin{aligned} & x_i^D - \sum_{j=1}^{n} (a_{ij}^D + m_{ij})\,\tilde{x}_j^D + g_i\,\tilde{x}_i^D - \sum_{k=1}^{n} h_{ik}\,x_k^M + (x_i^M - \sum_{j=1}^{n} x_{ij}^M) = (y_i^D + y_i^M) \\ & x_i^M + \hat{g}_i\,\tilde{x}_i^D - x_i^M \qquad - \hat{k}_i\,x_i^M = 0 \end{aligned}$$

where all the variables and parameters are as described in earlier *industry technology* model formulations. Re-writing Eq. (5.25), we have

$$(5.26)\quad \begin{pmatrix} X^D \\ X^M \end{pmatrix} = \begin{bmatrix} [(A^D + M) - \hat{g}] & (H - I + \hat{k}) \\ -\hat{g} & (I + \hat{k}) \end{bmatrix} \begin{pmatrix} \tilde{X}^D \\ X^M \end{pmatrix} + \begin{pmatrix} (Y^D + Y^M) \\ 0 \end{pmatrix}.$$

Again, replacing $\tilde{X}^D$ by $U\,X^D$, and following the same type of mathematical manipulations as described in Eqs. (4.17) through (4.22) in Chapter IV, we can re-write Eq. (5.26) as

$$(5.27)\quad \begin{bmatrix} \left\{ I - [(A^D + M) - \hat{g}]\ U \right\} & (I - \hat{k} + H) \\ \hat{g}\,U & -\hat{k} \end{bmatrix} \begin{pmatrix} X^D \\ X^M \end{pmatrix} = \begin{pmatrix} (Y^D + Y^M) \\ 0 \end{pmatrix}$$

which, after matrix inversion, becomes

$$(5.28)\quad \begin{pmatrix} X^D \\ \\ X^M \end{pmatrix} = \begin{bmatrix} \left[\left\{I - [(A^D + M) - \hat{g}]U\right\} + (I - \hat{k} + H)\hat{k}^{-1}\hat{g}U\right]^{-1} & \left[\left\{I - [(A^D + M) - \hat{g}]U\right\} + (I - \hat{k} + H)\hat{k}^{-1}\hat{g}U\right]^{-1}(I - \hat{k} + H)\hat{k}^{-1} \\ \\ \hat{k}^{-1}\hat{g}U\left[I - [(A^D + M) - \hat{g}]U + (I - \hat{k} + H)\hat{k}^{-1}\hat{g}U\right]^{-1} & \hat{k}^{-1} + \hat{k}^{-1}\hat{g}U\left[\left\{I - [(A^D + M) - \hat{g}]U\right\} + (I - \hat{k} + H)\hat{k}^{-1}\hat{g}U\right]^{-1}(I - \hat{k} + H)\hat{k}^{-1} \end{bmatrix} \begin{pmatrix} Y^D + Y^M \\ \\ 0 \end{pmatrix}$$

where the four submatrices have the same economic interpretation as before. The differences between the *commodity technology* and *industry technology* versions of Alternative IV can be seen by comparing the four submatrices in Eq. (5.28) with those given in Eq. (5.24).

5. Alternative V

In all of the earlier extensions, noncompetitive imports were assumed to be treated exogenously. This may not be entirely desirable in some instances when the particular regional or national economy under consideration is relatively small and, further, when the intended model contains considerable commodity detail. It will be generally observed for regional or metropolitan area economies that the finer the sectoral or commodity detail, the more noncompetitive imports there will be. Thus, in order to develop a *more* thorough understanding of the economic structure of a large or small region for which a fairly detailed model is contemplated, it would be desirable to design the model so that noncompetitive imports can be treated endogenously.

With the explicit introduction of noncompetitive imports as endogenous variables in the system, new options arise in model design. *If,* for example, the *technology* matrix is observable only as a *combined* matrix (i.e., the matrices A^D and M are observable only as a single, *collapsed* matrix), final demand for *regionally* produced products, competitive imports, and noncompetitive imports are observable (and exogenously predictable) as three separate vectors, and finally, noncompetitive imports used for intermediate consumption are known only in the aggregate by commodity on the supply side but not by each consuming sector (i.e., a noncompetitive imports input matrix is not observable), *then* the model design would be different from what would have resulted under an alternative set of conditions.

Under the particular set of conditions just described, a further extension of both the *commodity technology* model and the *industry technology* model can be developed. Other extensions are possible but they will not be take up here.

a. Alternative V: "Commodity Technology" Model

The commodity flow balance equations of the *commodity technology* model can be written as

(5.29)

$$
\begin{array}{lllll}
x_i^D - \sum_{j=1}^{n} (a_{ij}^D + m_{ij})\, x_j^D + g_i\, x_i^D - \sum_{k=1}^{n} h_{ik}\, x_k^M - \sum_{k=1}^{n} \underset{\sim}{h}_{ik} x_k^R & & & = & y_i^D \\
- g_i\, x_i^D & + x_i^M & & = & y_i^M \\
- r_i\, x_i^D & & + x_i^R & = & y_i^R
\end{array}
$$

where x_i^R is total noncompetitive imports of product i;

$\underset{\sim}{h}_{ik}$ has the same interpretation as h_{ik} but refers to noncompetitive imports;

$r_i = \dfrac{\sum_{j=1}^{n} x_{ij}^R}{x_i^D}$ is total noncompetitive imports of product i used for intermediate consumption as a proportion of total *regional* production of product i; and

y_i^R is total noncompetitive imports of product i used for final consumption;

$x_i^D = x_j^D$ for $i = j$; $x_i^M = x_k^M$ for $i = k$.

Eq. (5.29) can be re-written more compactly as

$$(5.30)\quad \begin{bmatrix} [I-(A^D+M)+\hat{g}] & -H & -\underset{\sim}{H} \\ -\hat{g} & I & 0 \\ -\hat{r} & 0 & I \end{bmatrix} \begin{pmatrix} X^D \\ X^M \\ X^R \end{pmatrix} = \begin{pmatrix} Y^D \\ Y^M \\ Y^R \end{pmatrix}$$

and, subsequently, as

$$(5.31)\quad \begin{pmatrix} X^D \\ X^M \\ X^R \end{pmatrix} = \begin{bmatrix} [I-(A^D+M)+\hat{g}] & -H & -\underset{\sim}{H} \\ -\hat{g} & I & 0 \\ -\hat{r} & 0 & I \end{bmatrix}^{-1} \begin{pmatrix} Y^D \\ Y^M \\ Y^R \end{pmatrix}$$

where $\hat{r}$ is an nxn diagonal matrix consisting of r_i and where all submatrices are of the same order.

Let the submatrices of the inverted parameter matrix be denoted as

$$(5.32)\quad L^{-1} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

After inverting the parameter matrix in Eq. (5.31) by following the mathematical procedure outlined in Appendix D, we find $B_{11}, B_{12}, \ldots, B_{33}$ as follows:

$$(5.33a)\quad B_{11} = \left\{[I-(A^D+M)+\hat{g}] - \underset{\sim}{H}\hat{r} - H\hat{g}^{-1}\right\}^{-1}$$

$$(5.33b) \qquad B_{12} = \left\{ [I - (A^D + M) + \hat{g}] - \underset{\sim}{H}\hat{r} - H\hat{g}^{-1} \right\}^{-1} H$$

$$(5.33c) \qquad B_{13} = \left\{ [I - (A^D + M) + \hat{g}] - \underset{\sim}{H}\hat{r} - H\hat{g}^{-1} \right\}^{-1} \underset{\sim}{H}$$

$$(5.33d) \qquad B_{21} = \hat{g} \left\{ [I - (A^D + M) + \hat{g}] - \underset{\sim}{H}\hat{r} - H\hat{g}^{-1} \right\}^{-1}$$

$$(5.33e) \qquad B_{22} = I + \hat{g} \left\{ [I - (A^D + M) + \hat{g}] - \underset{\sim}{H}\hat{r} - H\hat{g}^{-1} \right\}^{-1} H$$

$$(5.33f) \qquad B_{23} = \hat{g} \left\{ [I - (A^D + M) + \hat{g}] - \underset{\sim}{H}\hat{r} - H\hat{g}^{-1} \right\}^{-1} \underset{\sim}{H}$$

$$(5.33g) \qquad B_{31} = \hat{r} \left\{ [I - (A^D + M) + \hat{g}] - \underset{\sim}{H}\hat{r} - H\hat{g}^{-1} \right\}^{-1}$$

$$(5.33h) \qquad B_{32} = \hat{r} \left\{ [I - (A^D + M) + \hat{g}] - \underset{\sim}{H}\hat{r} - H\hat{g}^{-1} \right\}^{-1} H$$

$$(5.33i) \qquad B_{33} = I + \hat{r} \left\{ [I - (A^D + M) + \hat{g}] - \underset{\sim}{H}\hat{r} - H\hat{g}^{-1} \right\}^{-1} \underset{\sim}{H} \; .$$

The nine *multiplier* matrices can be given the following economic interpretations:

B_{11} is the key *multiplier* or inverse submatrix in which every element represents the direct and indirect *regional* output that is required of a given commodity per unit of final demand for the products of a given (column) sector. In short, B_{11} transforms a bill of goods for *regionally* produced products into direct and indirect *regional* output requirements for each product.

B_{12} indicates the direct and indirect contribution of competitive imports entering the system to *regional* production levels, through the *margin* or distribution services that are rendered by the economy in the process of delivering competitive imports to the local consumers.

B_{13} has the same interpretation as B_{12}, except that B_{13} refers to the contribution of noncompetitive imports entering the system.

B_{21} represents the direct and indirect effects of a unit shift in final demand for *locally* produced products *on* the competitive imports requirements of the economy. It thus helps us determine the direct and indirect competitive imports requirements of the economy for intermediate consumption, corresponding to a given bill of goods for locally produced products.

B_{22} and B_{23} respectively indicate how much of a decrease in the *regional* production of each commodity will result from a shift in final demand from *regional* products to competitive imports on the one hand, and from a shift in final demand from *regional* products to non-competitive imports, on the other.

B_{31} helps us determine the direct and indirect noncompetitive imports requirements of the economy, corresponding to a given bill of goods for *regionally* produced products.

Finally, B_{32} and B_{33} represent, respectively, the direct and indirect effects of final demand for competitive and noncompetitive imports *on* the noncompetitive imports requirements of the economy.

b. Alternative V: "Industry Technology" Model

The *industry technology* version of Extension 5 can be developed by writing the commodity flow balance equations as

$$
\text{(5.34)} \qquad
\begin{array}{llllll}
x_i^D - \sum_{j=1}^{n} (a_{ij}^D + m_{ij})\,\tilde{x}_j^D + g_i\,\tilde{x}_i^D & - \sum_{k=1}^{n} h_{ik}\,x_k^M & - \sum_{k=1}^{n} \underset{\sim}{h}_{ik}\,x_k^R & = y_i^D \\[2ex]
- g_i\,\tilde{x}_i^D & + x_i^M & & = y_i^M \\[2ex]
- r_i\,\tilde{x}_i^D & & + x_i^R & = y_i^R
\end{array}
$$

where the terms are as defined in Eq. (5.29) and where

$$
\tilde{x}_i^D = \tilde{x}_j^D \ \text{ for } i = j; \quad x_i^M = x_k^M \quad \text{for } i = k;
$$

$$
g_i = \frac{\sum_{j=1}^{n} x_{ij}^M}{\tilde{x}_i^D} \quad ; \quad r_i = \frac{\sum_{j=1}^{n} x_{ij}^R}{\tilde{x}_i^D} \quad .
$$

Eq. (5.34) can be re-written as follows:

$$(5.35)\quad \begin{pmatrix} X^D \\ X^M \\ X^R \end{pmatrix} = \begin{bmatrix} [(A^D + M) - \hat{g}] & H & \underset{\sim}{H} \\ \hat{g} & 0 & 0 \\ \hat{r} & 0 & 0 \end{bmatrix} \begin{pmatrix} \tilde{X}^D \\ X^M \\ X^R \end{pmatrix} + \begin{pmatrix} Y^D \\ Y^M \\ Y^R \end{pmatrix}$$

where $\hat{g}$ and $\hat{r}$ are diagonal matrices consisting, respectively, of g_i and r_i. Again, replacing the local *industry* output vector $\tilde{X}^D$ by its equivalent, $U\,X^D$, we have

$$(5.36)\quad \begin{pmatrix} X^D \\ X^M \\ X^R \end{pmatrix} = \begin{bmatrix} [(A^D + M) - \hat{g}]\ U & H & \underset{\sim}{H} \\ \hat{g}\,U & 0 & 0 \\ \hat{r}\,U & 0 & 0 \end{bmatrix} \begin{pmatrix} X^D \\ X^M \\ X^R \end{pmatrix} + \begin{pmatrix} Y^D \\ Y^M \\ Y^R \end{pmatrix}$$

which, following the same type of mathematical manipulations as described in Eqs. (4.17) through (4.22) in Chapter IV, can be re-written as

$$(5.37)\quad \begin{bmatrix} \{I - [(A^D + M) - \hat{g}]\ U\} & -H & -\underset{\sim}{H} \\ -\hat{g}\,U & I & 0 \\ -\hat{r}\,U & 0 & I \end{bmatrix}^{-1} \begin{pmatrix} X^D \\ X^M \\ X^R \end{pmatrix} = \begin{pmatrix} Y^D \\ Y^M \\ Y^R \end{pmatrix}.$$

Again, let the submatrices of the inverted parameter matrix be denoted as

$$L^{-1} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

After inverting the parameter matrix in Eq. (5.37) by following the mathematical procedure outlined in Appendix D, we find $B_{11}, B_{12}, \ldots, B_{33}$ as follows:

$$(5.38a) \quad B_{11} = \left\{ I - [(A^D + M) - \hat{g}]\, U - \underset{\sim}{H}\,\hat{r}\,U - H\,\hat{g}\,U \right\}^{-1}$$

$$(5.38b) \quad B_{12} = \left\{ I - [(A^D + M) - \hat{g}]\, U - \underset{\sim}{H}\,\hat{r}\,U - H\,\hat{g}\,U \right\}^{-1} H$$

$$(5.38c) \quad B_{13} = \left\{ I - [(A^D + M) - \hat{g}]\, U - \underset{\sim}{H}\,\hat{r}\,U - H\,\hat{g}\,U \right\}^{-1} H$$

$$(5.38d) \quad B_{21} = \hat{g}\,U \left\{ I - [(A^D + M) - \hat{g}]\, U - \underset{\sim}{H}\,\hat{r}\,U - H\,\hat{g}\,U \right\}^{-1}$$

$$(5.38e) \quad B_{22} = I + \hat{g}\,U \left\{ I - [(A^D + M) - \hat{g}]\, U - \underset{\sim}{H}\,\hat{r}\,U - H\,\hat{g}\,U \right\}^{-1} H$$

$$(5.38f) \quad B_{23} = \hat{g}\,U \left\{ I - [(A^D + M) - \hat{g}]\, U - \underset{\sim}{H}\,\hat{r}\,U - H\,\hat{g}\,U \right\}^{-1} \underset{\sim}{H}$$

$$(5.38g) \quad B_{31} = \hat{r}\,U \left\{ I - [(A^D + M) - \hat{g}]\, U - \underset{\sim}{H}\,\hat{r}\,U - H\,\hat{g}\,U \right\}^{-1}$$

$$(5.38h) \quad B_{32} = \hat{r}\,U \left\{ I - [(A^D + M) - \hat{g}]\, U - \underset{\sim}{H}\,\hat{r}\,U - H\,\hat{g}\,U \right\}^{-1} H$$

$$(5.38i) \quad B_{33} = I + \hat{r}\,U \left\{ I - [(A^D + M) - \hat{g}]\, U - \underset{\sim}{H}\,\hat{r}\,U - H\,\hat{g}\,U \right\}^{-1} \underset{\sim}{H}$$

where the submatrices $B_{11}, B_{12}, \ldots, B_{33}$ have the same economic interpretation as in Eqs. (5.33a) through (5.33i). The differences between the two sets of results can be easily seen through a direct comparison.

C. ALTERNATIVE MODEL DESIGNS FOR MULTIREGIONAL ECONOMIC ANALYSIS

In regional input-output analysis, emphasis is placed on the structural interdependence of the economic sectors within a particular region, as well as on the economic relationships of the region with *the rest of the world.* By contrast, in multiregional analysis, emphasis is placed not only on the structural interdependence of the economic sectors within each respective region, but also, and more importantly, on the economic interdependence of the regions explicitly included in the study. Greater insight can be developed into the economic structure of a particular region when the network of economic relationships linking it to other regions are explicitly taken into account through multiregional models.

In the past, a variety of multiregional models have been developed, including Isard's interregional model,[17] Leontief's early *balanced* or *centrally connected* intranational model,[18] the interregional system given by Moses,[19] and finally the Leontief-Strout *gravity* model.[20]

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[17] Walter Isard, "Interregional and Regional Input-Output Analysis: A Model of a Space-Economy," *The Review of Economics and Statistics,* XXXIII (November, 1951), 318-329.

[18] Wassily Leontief, *et al., Studies in the Structure of the American Economy* (New York: Oxford University Press, 1953), 93-116.

Also see Wassily Leontief, *et al.,* "The Economic Impact - Industrial and Regional - of an Arms Cut," in Wassily Leontief, *Input-Output Economics* (New York: Oxford University Press, 1966), 184-222.

[19] Leon Moses, "A General Equilibrium Model of Production, Interregional Trade and Location of Industry," *The Review of Economics and Statistics,* XLII (November, 1960), 373-397; and his earlier article, "The Stability of Interregional Trading Patterns and Input-Output Analysis," *The American Economic Review,* XLV, 5 (December, 1955), 803-833.

[20] Wassily Leontief and Alan M. Strout, "Multiregional Input-Output Analysis," in Tibor Barna (ed.), *Structural Interdependence and Economic Development,* Proceedings of an International Conference on Input-Output Techniques, Geneva, September, 1961 (London: MacMillan and Co., Ltd., and New York: St. Martin's Press, 1963), 119-151.

Also see the chapter entitled "Multiregional Input-Output Analysis" in Wassily Leontief, *Input-Output Economics (New York: Oxford University Press, 1966), pp. 223-257*

We can also include in this list the model used by Chenery in a study of the North–South interdependence of the Italian economy,[21] and the model employed in Wonnacott's study of the interrelationship between Canada and the United States.[22] A review of multiregional systems, focusing mostly on Isard's model, can be found in a recent article by Ponsard.[23] His use of *transition graphs* to describe interregional flows makes his discussion particularly helpful. A more extensive review of multiregional systems can be found in a recent, unpublished doctoral dissertation by Davis.[24] Lastly, in a recent article Miller has examined in considerable detail the structure of models for interregional analysis.[25]

Theoretical or applied work on input–output models for interregional analysis in the past has been deficient for a number of reasons. *First,* no systematic exploration has been made of alternative model designs based upon alternative sets of conditions on data availability. The *type* of systematic exploration suggested here is represented by the five alternative model designs developed earlier in this chapter for the one–region case. *Secondly,* past interregional models or analyses have failed to capture interregional feedback effects in sufficient detail. No distinction, for example, has been drawn between *competitive* and

[21]Hollis B. Chenery, Paul G. Clark, and Vera Cao-Pinna, *The Structure and Growth of the Italian Economy* (Rome: U. S. Mutual Security Agency, 1953).

Also see, Hollis B. Chenery, "Interregional and International Input-Output Analysis," in Tibor Barna (ed.), *The Structural Interdependence of the Economy,* Proceedings of an International Conference on Input-Output Analysis, Varenna, 1954 (New York: John Wiley and Sons, Inc., 1956), 51-103.

[22] R. J. Wonnacott, Canadian-American Dependency, *An Interindustry Analysis of Production and Prices* (Amsterdam: North-Holland Publishing Co., 1961).

[23] Claude Ponsard, "Essai d'interprétation topologique des systèmes interrégionaux," *Revue Economique,* XVIII, 3 (Mai, 1967), 353-373.

[24]Harrell C. Davis, "A Multiregional Input-Output Model of the Western States Emphasizing Heavy Water-Using Sectors" (unpublished Ph.D. thesis, University of California, Berkeley, 1967), 128-153.

[25] Ronald E. Miller, "Interregional Feedback Effects in Input-Output Models: Some Preliminary Results," *Regional Science Association Papers,* XVII (1966), 105-125.

noncompetitive interregional trade flows, although their feedback effects would prove to be quite different. *Finally,* practically no explanation is provided in the literature on the empirical nature of the *technology* matrices used in interregional analysis. The criticisms in Chapters II and III directed at the *traditional* measurement of the *technological* coefficients matrices are equally applicable here. It is again necessary to point out that the mathematical and empirical structure of an interregional system would be different, depending on the *technology* assumption inherent in the model (i.e., *commodity technology* or *industry technology*).

In light of these observations, the remainder of this chapter will be devoted, *first,* to a brief review of the formal properties of selected interregional models and, *second*, to illustrate how alternative model designs can be developed by extending the five alternative model designs discussed earlier for the one-region case. For illustrative purposes, only the *commodity technology* model version of Alternative V is extended into the simplest two-region case, which is easily generalizable into the n-region case. Without any difficulty, the *industry technology* model version of Alternative V, as well as each one of the four alternative pairs of *commodity technology* and *industry technology* models developed earlier can be extended for multiregional analysis.

1. A Brief Review of Selected Interregional Models

To develop a more detailed understanding of the structure of interregional models formulated in the past and to gain a better appreciation of their strengths, as well as their limitations, a few of the more well known interregional models will be briefly reviewed. These consist of Isard's interregional model, Leontief's *balanced* or *centrally connected* model, and the interregional model given by Moses. A comprehensive review of the

Leontief–Strout *gravity* model, which is omitted from the discussion here, can be found in Davis.[26]

a. Isard's Interregional Model

Isard's interregional system can be mathematically stated as follows:

$$(5.39)\qquad {}_k x_i - \sum_{\ell=1}^{n} \sum_{j=1}^{n} {}_{k\ell}a_{ij}\ {}_\ell x_j = {}_k y_i \qquad \begin{cases} i, j = 1,2,\ldots,m \\ k, \ell = 1,2,\ldots,n \end{cases}$$

where ${}_k x_i$ is the total output of product i produced by region k, ${}_k x_i = {}_k x_j$ for $i = j$; ${}_k y_i$ is the total final demand in region k for the regionally produced product i; and,

${}_{k\ell}a_{ij} = {}_{k\ell}x_{ij}/{}_\ell x_j$, where ${}_{k\ell}x_{ij}$ is the total amount of product i produced by region k that is required by sector j in region ℓ.

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[26] Davis, *op. cit.*, pp. 147-153.

The basic structural relationships in the Leontief-Strout system for interregional flows are given by the equation

$$X_{i.gh} = \frac{X_{i.go}\ X_{i.oh}}{X_{i.00}} \cdot q_{i.gh}$$

which states that the flow of commodity i from region g to region h is directly proportional to the total production of commodity i in region g ($X_{i.go}$), the consumption of commodity i in region h ($X_{i.oh}$), a set of variables collectively denoted by $q_{i.gh}$, and inversely proportional to the total production of commodity i in all regions. The term, $q_{i.gh}$, is expressed as

$$q_{i.gh} = (c_{i.g} + k_{i.h})\ d_{i.gh}\ \delta_{i.gh}$$

where $d_{i.gh}$ is a measure of the inverse of the transport cost of moving a unit of commodity i from region g to region h; $\delta_{i.gh}$ is a parameter designed to limit the number of variables in the system (i.e., if, for any of a variety of economic reasons no export of commodity i from region g to region h is expected, $\delta_{i.gh} = 0$; otherwise $\delta_{i.gh} = 1$); and $c_{i.g}$ and $k_{i.h}$ are two additional parameters that characterize the relative position of region g vis-à-vis all other regions as a supplier, and of region h as a user, of good i.

For further details, see Leontief and Strout, *loc. cit.*, and Davis, *loc. cit.*

Re-writing Eq. (5.39), we have the somewhat familiar formulation

$$(5.40) \qquad {}_kX = [I - \sum_{\ell=1}^{n} \sum_{j=1}^{m} {}_{k\ell}A_{ij}]^{-1} \quad {}_kY$$

for the special case where $k = \ell$.

In Isard's system, the possibility of substitutability among physically similar but geographically different commodities is not allowed. Further, no distinction is drawn between competitive and noncompetitive interregional flows. It is assumed that, for each region, there exists a separate, regional *technology* matrix which reflects only the input requirements for locally produced products. Interregional flows are shown in separate trade matrices.

The major difficulty with Isard's system is that its information requirements are nearly impossible to meet in practice. Even if the model's information requirements are met, there still remain the question of the treatment of interregional flows in this particular way. The lack of distinction between competitive and noncompetitive imports into a given region cannot be ignored. Further, the system does not account for interregional flows that are used for final consumption; at least the formulation does not explicitly deal with this problem.

b. Leontief's "Balanced" or "Centrally Connected" Model

In his *balanced* or *centrally connected* model, Leontief assumed, *first*, that a given sector has the same cost or input structure in each region, *second* that each commodity can be classified as either regional or national and that the former are not subject to interregional trade but are balanced at the regional level, and *thirdly*, that the same sectors in different regions which produce a commodity which is subject to interregional trade contribute to the total output of that commodity in fixed proportions.

The model can be formulated as follows. Let regional commodities be denoted by the subscript r and national commodities by the subscript s. Then, for the economy as a whole

$$(5.41) \qquad \begin{pmatrix} X_r \\ X_s \end{pmatrix} = \begin{bmatrix} A_{rr} & A_{rs} \\ A_{sr} & A_{ss} \end{bmatrix} \begin{pmatrix} X_r \\ X_s \end{pmatrix} + \begin{pmatrix} Y_r \\ Y_s \end{pmatrix}$$

where X_r and X_s are the output vectors, Y_r and Y_s are the final demand vectors, and the A's are the *technological* coefficients matrices.

Re-writing Eq. (5.41), we have the standard formulation

$$\text{(5.42)} \qquad \begin{bmatrix} (I - A_{rr}) & -A_{rs} \\ -A_{sr} & (I - A_{ss}) \end{bmatrix} \begin{pmatrix} X_r \\ X_s \end{pmatrix} = \begin{pmatrix} Y_r \\ Y_s \end{pmatrix}$$

which, after matrix inversion, becomes

$$\text{(5.43)} \qquad \begin{pmatrix} X_r \\ X_s \end{pmatrix} = \begin{bmatrix} [(I - A_{rr}) - A_{rs}(I - A_{ss})^{-1} A_{sr}]^{-1} & [(I - A_{rr}) - A_{rs}(I - A_{ss})^{-1} A_{sr}]^{-1} A_{rs}(I - A_{ss})^{-1} \\ (I - A_{ss})^{-1} A_{sr} [(I - A_{rr}) - A_{rs}(I - A_{ss})^{-1} A_{sr}]^{-1} & (I - A_{ss})^{-1} + (I - A_{ss})^{-1} A_{sr} [(I - A_{rr}) - A_{rs}(I - A_{ss})^{-1} A_{sr}]^{-1} A_{rs}(I - A_{ss}) \end{bmatrix} \begin{pmatrix} Y_r \\ Y_s \end{pmatrix} .$$

Let a region be denoted by the subscript k placed in front of a vector and let the fixed proportions in which region k contributes to the total output of commodities entering into interregional trade be denoted by the diagonal matrix ${}_kE = ({}_ke)$. Then, for region k

$$\text{(5.44)} \qquad {}_kX_s = {}_kE_s\, X_s .$$

When we substitute for X_s its equivalent from Eq. (5.43), we have

$$(5.45) \quad {}_kX_s = {}_kE_s \Big\{ (I - A_{ss})^{-1} A_{sr}[(I - A_{rr}) - A_{rs}(I - A_{ss})^{-1} A_{sr}]^{-1} Y_r$$

$$+ (I - A_{ss})^{-1} Y_s$$

$$+ (I - A_{ss})^{-1} A_{sr} [(I - A_{rr}) - A_{rs}(I - A_{ss})^{-1} A_{sr}]^{-1} A_{rs}(I - A_{ss}) Y_s \Big\} ,$$

which shows how the output levels of the national commodities produced in a given region are determined in the model if final demand for *regional* commodities is known in each region and if final demand for national commodities is known for all regions combined.

Similarly, the output levels of *regional* commodities produced in a given region are determined in the model as follows:

$$(5.46) \quad {}_kX_r = [(I - A_{rr}) - A_{rs}(I - A_{ss})^{-1} A_{sr}]^{-1} Y_r$$

$$+ [(I - A_{rr}) - A_{rs}(I - A_{ss})^{-1} A_{sr}]^{-1} A_{rs}(I - A_{ss})^{-1} Y_s .$$

If final demand for national commodities is known for each region separately. then a trading pattern will emerge for each region as the balance between the region's output and consumption of national commodities. In this system, the source of imports or the destination of the exports is not shown. In other words, the regions are not directly linked in the system but only in a remote sense through the economy as a whole.

As an *intranational* rather than an interregional model, Leontief's *balanced* or *centrally connected* system provides a quick and easy way of using a national *technological* coefficients matrix in analyzing, for example, the regional repercussions of such national policies as a cutback in military defense expenditures. The model clearly leaves a great deal to be desired, particularly because of its assumption of a homogeneous input structure for the same sector in each region, because of the built-in ambiguities surrounding the delineation of commodities as *regional* or *national*, and finally because of its *nontreatment* of interregional economic relationships.

c. The Interregional Model of Moses:

The three-region interregional model given by Moses can be generalized into n industries (commodities) and r regions. In the notations below, regions are denoted as superscripts and industries (commodities) as subscripts.

The Moses model requires two basic structural matrices for each region. The *first*, $A^s = [a^s_{ij}]$, represents an nxn *technological* coefficients matrix for a given region s where the underlying intersectoral transactions matrix (apparently) includes not only the flow of regionally produced goods but also the flow of competitive and noncompetitive imports used for intermediate consumption. No explanation is provided on the empirical nature of the *technology* matrices. In the model, r separate A^s matrices are required for each of the r regions.

The *second* matrix, $T_k = [t^{pq}_k]$, is an r xr matrix in which each element t^{pq}_k shows the proportion of region q's purchases of a particular product k which originates in region p. A separate T_k matrix is required for each commodity, so that there are n such matrices.

Both the A^s and the T_k matrices can be written as block-diagonal matrices as follows:

$$(5.47) \qquad A = \begin{bmatrix} A^1 & 0 & \dots & 0 \\ 0 & A^2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A^r \end{bmatrix} ; \; T = \begin{bmatrix} T_1 & 0 & \dots & 0 \\ 0 & T_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & T_n \end{bmatrix}$$

where both A and T are of the order nr xnr.

Correspondingly, we can define the three sets of fundamental variables involved in the model as follows:

y^s_k : the final demand for commodity k by region s (regardless of the source of supply);

d^s_k : the total demand for commodity k by region s;

x^s_k : the total output of commodity k by region s.

In more compact terms, we have the following vectors in the system:

$$Y_c = \begin{pmatrix} Y^1 \\ \cdot \\ \cdot \\ \cdot \\ Y^r \end{pmatrix}, \quad D_c = \begin{pmatrix} D^1 \\ \cdot \\ \cdot \\ \cdot \\ D^r \end{pmatrix}, \quad X_c = \begin{pmatrix} X^1 \\ \cdot \\ \cdot \\ \cdot \\ X^r \end{pmatrix} \tag{5.48}$$

$$\begin{matrix} nr \times 1 & \qquad & nr \times 1 & \qquad & nr \times 1 \end{matrix}$$

and

$$Y_w = \begin{pmatrix} Y_1 \\ \cdot \\ \cdot \\ \cdot \\ Y_n \end{pmatrix}, \quad D_w = \begin{pmatrix} D_1 \\ \cdot \\ \cdot \\ \cdot \\ D_n \end{pmatrix}, \quad X_w = \begin{pmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{pmatrix} \tag{5.49}$$

$$\begin{matrix} nr \times 1 & \qquad & nr \times 1 & \qquad & nr \times 1 \end{matrix}$$

where the subscript c denotes ordering by regions and the subscript w denotes ordering by commodities. Thus, for example, Y^1 represents total final demand for n products by region 1, while Y_1 represents total final demand for product 1 by each of the r regions. Since each of the orderings Y_w, = D_w, and X_w is simply a rearrangement of the terms in the corresponding ordering by regions, there exists a permutation matrix, E, of order nrxnr, which will convert one into the other:

$$Y_c = E\,Y_w \quad ; \quad D_c = E\,D_w \quad ; \quad X_c = E\,X_w\,. \tag{5.50}$$

Next, we can write the two sets of structural equations that make up the model:

(5.51)
$$D^s = A^s X^s + Y^s \qquad s = 1,2,...,r$$

$$X_k = T_k D_k \qquad k = 1,2,...,n,$$

where the *first* equation states that the total demand of a region for each regionally produced or imported good is the sum of its final demand, Y^s, and its intermediate demand, $A^s X^s$. If the total demand, D_k, for each commodity in each region is known, and if regions acquire each good according to a fixed pattern of interregional trade, then the interregional flows of goods and services can be determined by the *second* equation which states the equality of demand ($T_k D_k$) and supply (X_k).

When final demands, Y^s, have been stipulated, this system can be solved for outputs, X^s, by eliminating the variables D^s and D_k from Eq. (5.51). In order to do this, we first have to re-write Eq. (5.51) as

(5.52)
$$D_c = A X_c + Y_c$$

$$X_w = T D_w$$

where all the terms are as defined earlier.

The second equation, $X_w = T D_w$, can be re-written as $X_w = T E^{-1} E D_w$. Upon premultiplication by E we have

(5.53)
$$E X_w = E T E^{-1} E D_w$$

or

(5.54) $$X_c = T^* D_c$$

where $$T^* = E\ T\ E^{-1},$$

and

$$E\ X_w = X_c;\ E\ Y_w = Y_c \qquad \text{from Eq. (5.50).}$$

When we now premultiply the first equation in Eq. (5.52) by T^*, we have

(5.55) $$T^* D_c = T^* A\ X_c + T^* Y_c$$

which is the same as

(5.56) $$X_c = T^* A X_c + T^* Y_c$$

since $$T^* D_c = X_c \qquad \text{from Eq. (5.54).}$$

Eq. (5.56) can be re–written as

(5.57) $$[I - T^* A]\ X_c = T^* Y_c$$

and finally as

(5.58) $$X_c = [I - T^* A]^{-1}\ T^* Y_c.$$

When the demands by the final demand sectors in each region are given, the system first converts these demands into a set of shipments on final demand account, $T^* Y_c$,by each region, and then determines all regional outputs.

The formal properties of the model consist of the exposition given above. A dynamic version of the model can be found in an Appendix by Fei and Moses.[27] The basic static model described here has fixed trading patterns for all goods but no restrictions on relative regional outputs. It thus contrasts with Leontief's *balanced* model which allows variable regional outputs but no trade in regionally balanced goods and which further allows variable trading patterns but fixed relative regional outputs entering into regional trade.

d. Summary Remarks on Interregional Models

As pointed out earlier, the interregional models developed in the past suffer from at least two basic deficiencies. *First,* no distinction is made in these models between competitive and noncompetitive imports, although such a distinction has been a standard practice in input-output analysis at the national level for a long time. Further, such a distinction is necessary in interregional analysis, since the interregional feedback effects of shifts in each region's final demand for competitive or noncompetitive imports would be quite different and would have different policy implications for each respective region. *Secondly,* the *technology* assumption in these models is left completely undefined and unclear. One could only presume that the *technology* matrices assumed or used in these models are of the traditional type, which has been severely criticized in this dissertation. Finally, these models appear to have been developed on an *ad hoc* basis, without a systematic examination of possible alternatives tailored to alternative sets of conditions on data availability and on the manner in which the *technology* matrices are observable. The illustrative two-region extension given here is suggestive of other extensions that can be developed from the one-region models presented earlier. This would be a positive step in the direction of opening up new options and possibilities in model construction for multiregional economic analysis.

2. An Illustrative Two-Region Extension of Alternative V

To illustrate how the five alternative sets of *commodity technology* and *industry technology* models developed earlier for the one-region case (i.e., for national or regional economic analysis) can be extended for multiregional economic analysis, the *commodity technology* model under Alternative V will be extended into the two-region case. It will be

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[27] John C. H. Fei and Leon N. Moses, "Appendix" to the article by Moses entitled "The Stability of Interregional Trading Patterns and Input-Output Analysis," *The American Economic Review,* XLV, 5 (December, 1955), 827-832.

seen that, in principle, the two–region extension can be easily generalized into the n–region case. Multiregional extensions of the type explored here should bring greater flexibility, clarity, and thoroughness to the analysis of the structural interdependence of economically linked regions.

A two–region *commodity technology* model extension of Alternative V can be developed by first underlying the following points. *First*, two trading *regions*, which may or may not be geographically contiguous, are identified. The *rest of the world* is identified as a third *region*, with which either of the two linked *regions* is assumed to have an economic relationship. Notationally, superscripts are used to denote the respective regions, the first digit standing for the producing or origin *region* and the second representing the consuming or destination *region. Secondly,* it is assumed, as under Alternative V, that in each of the two linked *regions,* the *technology* matrix is observable only as a composite matrix, rather than as three separate matrices (i.e., A^{11}, M^{21}, and M^{31} are not separately observable). *Thirdly,* it is assumed that information is available on the interregional flow of competitive, as well as noncompetitive, imports used by each region for both intermediate and final consumption.

Skipping the usual commodity flow balance equations that characteristically describe the basic accounting relationships in a model, we can more succinctly and without any conceptual loss state the fundamental two–region simultaneous equations system as follows:

(5.59)

$$
\left[\begin{array}{ccccc|ccccc}
[I-(A^{11}+M^{21}+M^{31})+\hat{g}^{21}+\hat{g}^{31}] & H^{21} & H^{31} & \underset{\sim}{H}^{21} & \underset{\sim}{H}^{31} & 0 & 0 & 0 & 0 & 0 \\
-\hat{g}^{21} & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\hat{g}^{31} & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\hat{r}^{21} & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
-\hat{r}^{31} & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & [I-(A^{22}+M^{12}+M^{32})+\hat{g}^{12}+\hat{g}^{32}] & H^{12} & H^{32} & \underset{\sim}{H}^{12} & \underset{\sim}{H}^{32} \\
0 & 0 & 0 & 0 & 0 & -\hat{g}^{12} & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\hat{g}^{32} & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\hat{r}^{12} & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & -\hat{r}^{32} & 0 & 0 & 0 & I
\end{array}\right]
\left(\begin{array}{c}
X^{1} \\ X^{21} \\ X^{31} \\ {}^{*}X^{21} \\ {}^{*}X^{31} \\ \hline X^{2} \\ X^{12} \\ X^{32} \\ {}^{*}X^{12} \\ {}^{*}X^{32}
\end{array}\right)
=
\left(\begin{array}{c}
Y^{11} \\ Y^{21} \\ Y^{31} \\ {}^{*}Y^{21} \\ {}^{*}Y^{31} \\ \hline Y^{22} \\ Y^{12} \\ Y^{32} \\ {}^{*}Y^{12} \\ {}^{*}Y^{32}
\end{array}\right)
$$

Where all vectors are of the order nxl and the submatrices are of the order nxn. The endogenous and exogenous variables (vectors) and the parameter submatrices can be described as follows:

Endogenous Variables (Vectors):

X^1 and X^2 denote the total output vectors for Region 1 and Region 2, respectively;

X^{21}, X^{31}, X^{12}, and X^{32} are vectors showing the interregional flow of total competitive imports, used for both intermediate and final consumption;

${}^*X^{21}$, ${}^*X^{31}$, ${}^*X^{12}$, and ${}^*X^{32}$ are vectors showing the interregional flow of noncompetitive imports of each product required in the region of destination, for both intermediate and final consumption;

Exogenous Variables (Vectors):

Y^{11} and Y^{22} denote the total final demand vectors for Region 1 and Region 2 for internally produced products;

Y^{21}, Y^{31}, Y^{22}, and Y^{12} are vectors showing interregional flows of competitive imports used for final consumption;

${}^*Y^{21}$, ${}^*Y^{31}$, ${}^*Y^{12}$, and ${}^*Y^{32}$ are vectors showing interregional flow of noncompetitive imports used for final consumption;

Parameters (Submatrices):

$(A^{11} + M^{21} + M^{31})$ and $(A^{22} + M^{12} + M^{32})$ are composite *regional technology* matrices for Region 1 and Region 2 (i.e., the three matrices are not observable separately), where

$$A^{11} = [a^{11}_{ij}] \quad , \quad a^{11}_{ij} = \frac{\sum_{j=1}^{n} x^{11}_{ij}}{x^1_j}$$

$$M^{21} = [m^{21}_{ij}] \quad , \quad m^{21}_{ij} = \frac{\sum_{j=1}^{n} x^{21}_{ij}}{x^1_i}$$

$$M^{31} = [m^{31}_{ij}] \quad , \quad m^{31}_{ij} = \frac{\sum_{j=1}^{n} x^{31}_{ij}}{x^1_i} \text{, etc.;}$$

H^{21}, H^{31}, H^{12}, H^{32} and $\underset{\sim}{H}^{21}$, $\underset{\sim}{H}^{31}$, $\underset{\sim}{H}^{12}$, $\underset{\sim}{H}^{32}$ are matrices, consisting of as many non-null rows as there are distributive sectors, showing the marginal propensity of the destination regions to provide distributive services in response to competitive and noncompetitive imports coming from the other region and from the *rest of the world.*

$\hat{g}^{21}$, $\hat{g}^{31}$, $\hat{g}^{12}$, and $\hat{g}^{32}$ are diagonal matrices consisting of

$$g^{21} = \frac{\sum_{j=1}^{n} x_{ij}^{21}}{x_i^1} \quad ; \qquad g^{31} = \frac{\sum_{j=1}^{n} x_{ij}^{31}}{x_i^1} \quad ; \qquad \text{etc.,}$$

showing the marginal (= average) propensity of Region 1 and Region 2, respectively, to require competitive imports of type i for intermediate consumption per unit of the regional (internal) production of the same product; and

$\hat{r}^{21}$, $\hat{r}^{31}$, $\hat{r}^{12}$, and $\hat{r}^{32}$ are diagonal matrices consisting of

$$r_i^{21} = \frac{\sum_{j=1}^{n} {}^*x_{ij}^{21}}{x_i^1} \quad ; \quad r_i^{31} = \frac{\sum_{j=1}^{n} {}^*x_{ij}^{31}}{x_i^1} \quad ; \quad \text{etc.,}$$

showing the marginal (= average) propensity of Region 1 and Region 2 to require noncompetitive imports of type i for intermediate consumption per unit of the regional output of product i.

In this particular treatment of noncompetitive imports, it is implicitly assumed that the total inflow of noncompetitive imports of type i from, for example, Region 2 to Region 1 is a linear function of the total output of product i in Region 1. For obvious reasons, this may prove to be a somewhat weak formulation, particularly if the receiving region's output of product-group i is almost nonexistent. It would be theoretically more defensible, in general, to assume that the inflow of noncompetitive imports of type i from, for example, Region 2 into Region 1 is a linear function of the latter's output of the particular product whose production requires the highest portion of the imported noncompetitive good of type i. In this case, we have

$$r_{ij}^{21} = \frac{\sum_{j=1}^{n} {}^*x_{ij}^{21}}{x_j^1} \quad ; \quad r_{ij}^{31} = \frac{\sum_{j=1}^{n} {}^*x_{ij}^{31}}{x_j^1} \quad ; \quad \text{etc.,}$$

where, for example, a given noncompetitive import may be viewed as a function of the output of any of the j=1,2,...,n regional sectors. For illustrative purposes, let us assume that Region 1 consists of four sectors, and that the inflow of noncompetitive imports from Region 2 into Region 1 is hypothetically defined by the following functional relationships:

$$\sum_{j=1}^{n} {}^*x_{1j}^{21} = f(x_4^1)$$

$$\sum_{j=1}^{n} {}^*x_{2j}^{21} = f(x_3^1)$$

$$\sum_{j=1}^{n} {}^*x_{3j}^{21} = f(x_1^1)$$

$$\sum_{j=1}^{n} {}^*x_{4j}^{21} = f(x_2^1)$$

which result in

$$r_1^{21} = \frac{\sum_{j=1}^{n} {}^*x_{1j}^{21}}{x_4^1} \quad ; \quad r_2^{21} = \frac{\sum_{j=1}^{n} {}^*x_{2j}^{21}}{x_3^1} \quad ;$$

$$r_3^{21} = \frac{\sum_{j=1}^{n} {}^*x_{3j}^{21}}{x_1^1} \quad ; \quad \text{and } r_4^{21} = \frac{\sum_{j=1}^{n} {}^*x_{4j}^{21}}{x_2^1} \quad .$$

We can write these results more compactly as

$$\begin{bmatrix} 0 & 0 & 0 & r_{14}^{21} \\ 0 & 0 & r_{23}^{21} & 0 \\ r_{31}^{21} & 0 & 0 & 0 \\ 0 & r_{42}^{21} & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \\ x_4^1 \end{pmatrix} = \begin{pmatrix} r_{14}^{21} \; x_4^1 \\ r_{23}^{21} \; x_3^1 \\ r_{31}^{21} \; x_1^1 \\ r_{42}^{21} \; x_2^1 \end{pmatrix} = \begin{pmatrix} {}^*\underset{\sim}{x}_1^{21} \\ {}^*\underset{\sim}{x}_2^{21} \\ {}^*x_3^{21} \\ {}^*\underset{\sim}{x}_4^{21} \end{pmatrix}$$

where $R^{21} = [r_{ij}^{21}]$ shows the marginal (= average) propensity of Region 1 to demand non-competitive imports from Region 2 for intermediate consumption,

$X^1 = (x_i^1)$ is the output vector for Region 1, and

${}^*\underset{\sim}{X}^{21} = ({}^*\underset{\sim}{x}_i^{21})$ is a vector showingsthe total noncompetitive imports input requirements of Region 1 from Region 2 for intermediate consumption.

It is clear that in our basic model formulation, each one of the diagonal matrices $\hat{r}^{21}$, $\hat{r}^{31}$, $\hat{r}^{12}$, and $\hat{r}^{32}$ can be replaced by the theoretically more desirable matrices R^{21}, R^{31}, R^{12}, and R^{32}, each one of which will have its particular arrangement of the r-coefficients, with a single r appearing in a given row or column, just as in the matrix $R^{21} = [r_{ij}^{21}]$ just described.

Referring back to the large, partitioned matrix in Eq. (5.59), we can see immediately that the *upper-right* and *lower-left* submatrices which are now null will remain to be null submatrices *after* the large, partitioned matrix is inverted. Further, the *upper-left* and *lower-right* submatrices in the inverse of the large parameter matrix in Eq. (5.59) will turn out to be the inverse, respectively, of the partitioned *upper-left* and *lower-right* submatrices. This can be seen in the following, *reduced form* statement of Eq. (5.59):

Eq. (5.60) enables us to study the structural interdependence of two regions not only with one another but also with the "rest of the world." The *upper-left* submatrix, after matrix inversion, helps us determine the output levels, and trade relationships of Region 1, given its final demand requirements for regionally produced products and for competitive and non-competitive imports from both Region 2 and from the "rest of the world." The *lower-right* submatrix, after matrix inversion, performs the same function for Region 2. Through the framework outlined here, it would be possible to trace the effects of a given shift, for example, in the final consumption demand of Region 1 for competitive imports from Region 2 (i.e., Y^{21}) *on* the economy of Region 1, as well as on the economy of Region 2. For illustrative purposes, let us suppose that every element in the vectory Y^{21} (i.e., Region 2's competitive imports requirements from Region 2 for final consumption). For Region 2, this would obviously mean an equal reduction in the vectory Y^{22} (i.e., Region 2's final demand requirements for regionally produced products). The direct and indirect effects of this drop in Y^{22} would be felt not only on Region 2's production levels but also on its trade with Region 1 and with the "rest of the world." For its part, and at least in the first round, Region 1 would experience an increase in either Y^{11} or Y^{21} or in both, following the initial drop in Y^{21}.

If the increase takes place in Y^{11}, Region 1's production levels will go up through import substitution, which in turn will affect its competitive and noncompetitive import requirements from Region 2 and from the "rest of the world."

(5.60)

$$
\begin{pmatrix} X^{1} \\ X^{21} \\ X^{31} \\ {}^{*}X^{21} \\ {}^{*}X^{31} \\ X^{2} \\ X^{12} \\ X^{32} \\ {}^{*}X^{12} \\ {}^{*}X^{32} \end{pmatrix}
=
\begin{bmatrix}
\begin{bmatrix}
[I-(A^{11}+M^{21}+M^{31})+\hat{g}^{21}+\hat{g}^{31}] & H^{21} & H^{31} & \underset{\sim}{H}^{21} & \underset{\sim}{H}^{31} \\
-\hat{g}^{21} & 0 & 0 & 0 & 0 \\
-\hat{g}^{31} & 0 & 0 & 0 & 0 \\
-R^{21} & 0 & 0 & 0 & 0 \\
-R^{31} & 0 & 0 & 0 & 0
\end{bmatrix}^{-1}
&
\begin{matrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{matrix}
\\
\begin{matrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{matrix}
&
\begin{bmatrix}
[I-(A^{22}+M^{12}+M^{32})+\hat{g}^{12}+\hat{g}^{32}] & H^{12} & H^{32} & \underset{\sim}{H}^{12} & \underset{\sim}{H}^{32} \\
-\hat{g}^{12} & 0 & 0 & 0 & 0 \\
-\hat{g}^{32} & 0 & 0 & 0 & 0 \\
-R^{12} & 0 & 0 & 0 & 0 \\
-R^{32} & 0 & 0 & 0 & 0
\end{bmatrix}^{-1}
\end{bmatrix}
\begin{pmatrix} Y^{11} \\ Y^{21} \\ Y^{31} \\ {}^{*}Y^{21} \\ {}^{*}Y^{31} \\ Y^{22} \\ Y^{12} \\ Y^{32} \\ {}^{*}Y^{12} \\ {}^{*}Y^{32} \end{pmatrix}
$$

The type of "comparative statics" interregional repercussion analysis as just described need not be confined to the first round effects alone. The *multiplier* or repercussion analysis can be extended into an examination of the second and subsequent round effects, as well. Furthermore, the framework just outlined can be used for interregional income and employment repercussions analysis, along similar lines as already suggested in Appendix B (refer to the discussion entitled "The Relationship between Final Demand, Sectoral Output Requirements, and Value Added (Income Generation)").

D. CONCLUDING REMARKS

The purpose of this chapter has been to continue the suggestions made at the end of Chapter II for input-output model formulation and to extend the conceptual work started in Chapter IV by presenting a systematic mathematical exploration of alternative model designs for national, regional, and multiregional economic analysis.

The alternative model formulations presented here cover the cases that are most likely to arise in actual practice. However, there still remain possibilities for further extension, under different assumptions on the treatment of noncompetitive imports.

The *commodity technology* and *industry technology* models developed under each of the five alternative sets of conditions for regional or national economic analysis should substantially increase the range of choice now available in input-output model construction and applications and should provide the conceptual clarity and mathematical thoroughness in input-output analysis that has been seriously lacking in the past. Likewise, the two-region extension given here for illustrative purposes should lead to the development of alternative model structures for multiregional analysis.

In conclusion, it is hoped that the work presented in this dissertation will spur further research on the quantitative analysis of problems related to economic interdependence, particularly as such research will contribute to the development of more enlightened public policy.

APPENDIX A

MATHEMATICAL PROPERTIES OF THE BASIC MODEL

A. INTRODUCTION

This appendix is essentially an extension of Chapter I. While Chapter I provides a rather brief but concise mathematical introduction to input-output models, the aim here is to go considerably beyond that introduction to present a comprehensive and unified treatment of a whole set of mathematical properties pertaining particularly to the basic open-static model. Unfortunately, in the input-output literature, such a unified treatment is entirely lacking. The presentation here draws heavily upon mathematical results on matrices that have been accumulated over many years, mostly in articles scattered over numerous foreign and domestic mathematical periodicals. The economics literature, to be sure, has developed its own reservoir of results over the years, but the contributions made by economists, intermittent at best, have been, on the whole, not quite satisfactory nor complete. In retrospect, the record of the economics literature on the subject could have been substantially better.

The organization of this appendix is as follows. First, it will be necessary to dispense a few basic definitions, both for completeness and also for the practical reason that they will be utilized rather frequently in subsequent discussions. These definitions concern positive, nonnegative, irreducible (indecomposable), and Minkowski-Leontief matrices, the fundamental algebraic eigenproblem, and the norms (moduli) of matrices.

Next, the discussion focuses on some known properties of positive and nonnegative matrices, omitting proofs. This discussion provides much of the mathematical background for the third section, where the mathematical properties of the *technological* **coefficients** matrix of the basic open model (i.e., the Minkowski-Leontief matrix) are presented.

The properties of M-matrices are then taken up, primarily to show how the Leontief matrix (I – A), comprises a special case of M-matrices (i.e., the general class of matrices with positive diagonals and non-positive off-diagonal elements), and, further, to demonstrate the mathematical properties of the Leontief matrix derived from the literature on M-matrices, as well as from the economics literature.

Lastly, it will be shown that many of the mathematical conditions that M-matrices in general, and the Leontief matrix in particular, must fulfill also comprise equivalent necessary and sufficient conditions for the existence of a unique nonnegative solution to the basic open input-output model. In this context, a mathematical demonstration and economic interpretation will be given of the positivity of the principal minors of the Leontief matrix, which underlies the well-known Hawkins-Simon condition(s) in economics. Following this, the convergence of the power (multiplier) series to the Leontief inverse will be proved, the economic meaning of the convergence process will be explained, and the *matrix multipliers* interpretation of the Leontief inverse will be explored.

The mathematical discussion in this chapter will concern itself only with algebraic results and proofs, omitting, for example, graph-theoretic treatments.[1] In order to keep the subject manageable, emphasis will be placed mostly on mathematical results rather than on detailed proofs, although certain proofs or mathematical digressions will be given where needed.

B. BASIC DEFINITIONS

1. Positive Matrices: A square matrix with real elements, $A = [a_{ij}]$ of order n $(i, j=1,2,\ldots,n)$, is called *positive* if all a_{ij} are strictly positive $(A > 0)$. We then have

$$M = \underset{i,j}{\mathrm{Max}}\ a_{ij} \geqslant a_{ij} \geqslant \underset{i,j}{\mathrm{Min}}\ a_{ij} = m > 0.$$

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[1] As an example, see David Rosenblatt, "On Linear Models and the Graphs of Minkowski-Leontief Matrices," *Econometrica,* XXV, 2(April, 1957), 325-338. In this paper, a Minkowski-Leontief matrix is viewed as a stochastic matrix, employing graph-theoretic concepts and formulations, in particular the concept of *cyclic net.* Of particular interest in this paper are the results obtained for non-stochastic Minkowski-Leontief matrices. As it will be clear later, it makes little algebraic sense to talk about *stochastic* Minkowski-Leontief matrices, since having each row sum equal unity, even after transposing the normal Minkowski-Leontief matrix, forces the maximal characteristic root (*Perron root*) of the matrix to be equal to unity. This, of course, will guarantee the singularity of both the Minkowski-Leontief matrix and the Leontief matrix, thus making the basic open model incapable of having any solution, let alone a unique nonnegative solution.

This definition, employed by Ostrowski[2] and many others, will be used here.

A slightly different definition is given by Karlin,[3] where, by a positive matrix, is meant the following: if $A \geqslant 0$ (i.e., A is a nonnegative matrix) and at least one a_{ij} is positive ($a_{ij} > 0$), then $A > 0$ is called a positive matrix. If every a_{ij} is positive, then the matrix is denoted as $\gg 0$.

An entirely different definition is strongly implied in a lemma due to McKenzie: "a necessary and sufficient condition for B [referring to a Leontief matrix] to be positive is that every principal minor of B have at least one column sum greater than zero."[4] As we shall see later the positivity of the principal minors of the Leontief matrix plays an important role in guaranteeing that any Leontief system defined by $X = (I - A)^{-1} Y$ has a unique nonnegative solution. But McKenzie's lemma is very misleading, in that what is actually at stake in this lemma is establishing the conditions for the positivity of the principal minors of the Leontief matrix and *not* the positivity of the Leontief matrix itself.

2. Nonnegative Matrices:[5] A square matrix with real elements, $A = [a_{ij}]$ of order n ($i, j = 1, 2, \ldots, n$), is said to be nonnegative or $A \geqslant 0$ if $a_{ij} \geqslant 0$ for each $i, j \in N$, where N is the set of indices $1, 2, \ldots, n$.

Unless otherwise specified, all matrices considered henceforth will have real elements. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order,

$$\begin{aligned}
&A \leqslant B \text{ if } a_{ij} \leqslant b_{ij} \text{ for all } i, j,\\
&A \leqslant B \text{ if } A \leqslant B \text{ and } A \neq B,\\
&B - A \geqslant 0,\\
&A < B \text{ if } a_{ij} < b_{ij} \quad \text{for all } i, j.
\end{aligned}$$[6]

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[2] See A.M. Ostrowski, "On Positive Matrices," *Mathematische Annalen,* CL (1963), 276. This definition is fairly standard in the literature. See, also, F.R. Gantmacher, *Applications of the Theory of Matrices,* translated and revised by J.L. Brenner, D.W. Bushaw, and S. Evanusa (New York: Interscience Publishers, Inc. 1959) p. 61.

[3] Samuel Karlin, *A First Course in Stochastic Processes* (New York: Academic Press, 1966), p. 475.

[4] Lionel McKenzie, "An Elementary Analysis in the Leontief System," *Econometrica,* XXV, 3 (July, 1957), 457.

[5] Gantmacher, *loc. cit.*

[6] Gerard Debreu and I.N. Herstein, "Nonnegative Square Matrices," *Econometrica,* XXI, 4 (October, 1953), 597.

The same definition and notations can be extended to vectors. Thus, for a vector $u = (\bar{u}_1, \bar{u}_2, ..., \bar{u}_n)$, $u \geqslant 0$ implies that $\bar{u}_i \geqslant 0$ $(i = 1,2,...,n)$; $u > 0$ implies that $\bar{u}_i > 0$. If u and v are two column vectors of the same order, for example, we can write $v \leqslant u$ if $u - v \geqslant 0$, etc. Clearly, $A \geqslant 0$ and $u \geqslant v$ imply $Au \geqslant Av$, while $A > 0$ and $u > v$ imply $Au > Av$.

3. Irreducible (Indecomposable) Matrices:[7] A nonnegative square matrix $A = [a_{ij}]$ of order $n\ (n \geqslant 2)$ is called *reduced* or *reducible* if there exists a permutation matrix P such that, through a simultaneous permutation of rows and columns, the matrix A can be transformed into the form

$$P'AP = PAP' = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

or into the form [8]

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where

P is the permutation matrix, a square matrix which in each row and each column has some one entry unity (one), others zero;

P′ is the transpose of P;

A_{11}, and square A_{22} are matrices and 0 is a zero matrix.

A *reduced* or *reducible* matrix can be characterized by the fact that there exist rows such that all elements which these k rows have in common with the first k columns equal zero. Alternatively, there exist k rows such that all elements which these k rows have in common with the remaining n − k columns equal zero.

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[7] *Ibid.* Also refer to Gantmacher, *loc. cit.;* Alston S. Householder, *The Theory of Matrices in Numerical Analysis* (New York: Blaisdell Publishing Co., A Division of Ginn and Co., 1964), p. 48; and Richard S. Varga, *Matrix Iterative Analysis* (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1965), p. 18.

[8] See Alfred Brauer, "A Method for the Computation of the Greatest Root of a Nonnegative Matrix," *Journal of the Society for Industrial and Applied Mathematics, Series B, Numerical Analysis* (SIAM J. NUMER, ANAL.), III, 4 (1966), 564.

If A is not *reduced* or *reducible*, then it is *irreducible, unreduced* or *indecomposable.*[9] The term irreducible (unzerlegbar) was introduced by Frobenius.[10] If A is a 1 x 1 complex matrix, then A is irreducible if its single entry is nonzero, and reducible otherwise.[11] A positive matrix is, by definition, irreducible.

4. Minkowski-Leontief Matrices:[12] A particular class of nonnegative square matrices, such as the matrix of technological coefficients in the open-static input-output model, are of the Minkowski-Leontief type if they have the following properties:

Let $A = [a_{ij}]$ denote a Minkowski-Leontief matrix. Then,

$$\text{(a)}\quad 0 \leqslant a_{ij} < 1 \qquad\qquad i,j = 1,2,\ldots,n.$$

$$\text{(b)}\quad \sum_{i=1}^{n} a_{ij} < 1$$

As it is by now abundantly clear, such matrices arise in connection with the solution of linear equations of the form

$$\text{(c)}\quad x_i = \sum_{j=1}^{n} a_{ij}\, x_j + y_i, \text{ or}$$

$$\text{(d)}\quad x_i - \sum_{j=1}^{n} a_{ij}\, x_j = y_i, \text{ or}$$

$$\text{(e)}\quad (\delta_{ij} - a_{ij})\, x_i = y_i, \text{ where } \delta \text{ is the kronecker delta}$$

which is unity if the subscripts are equal and zero otherwise. These equations represent equivalent statements of the familiar open-static input-output model.

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[9] Varga, *op. cit.* p. 19.

[10] G.Frobenius, "Über Matrizen aus nicht negativen Elementen," *Sitzungsberichte der königlich preussischen Akademie der Wissenschaften zu Berlin,* 26-27 (1912), 456-477. Also see V. Romanovsky, "Recherches sur les chains de Markoff," *Acta Mathematica,* LXVI (1936), 147-251; H. Geiringer, "On the Solution of Systems of Linear Equations by Certain Iterative Methods," in J.W. Edwards (ed.), *Reissner Anniversary Volume* (Ann Arbor, Michigan: University of Michigan Press, 1949), pp. 365-393; and H. Wielandt, "Unzerlegbare, nicht negative Matrizen," *Mathematische Zeitschrift,* LII (March, 1950), 642-648.

[11] Varga, *op. cit.,* p. 19.

[12] Minkowski proved that if $a_{ij} \geqslant 0$ and $s_j = \sum_{i=1}^{n} a_{ij} < 1$ $(i,j=1,2,\ldots,n)$ then $\det(I-A) = |I-A| > 0$. See, H. Minkowski, "Zur Theorie der Einheiten in den algebraischen Zahlkörpern," *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse* (1900), 90-93.

The strict inequalities that characterize the two conditions (a) and (b) presented here are often allowed to hold as equalities in the literature, for example in Bellman,[13] Wong,[14] Wong and Morgenstern.[15] Thus, conditions (a) and (b) are given as:

(f) $0 \leqslant a_{ij} \leqslant 1,$

(g) $\sum_{i=1}^{n} a_{ij} \leqslant 1,$

to which are usually added

(h) $a_{jj} < 1$ or $a_{ii} < 1,$

which means that every diagonal element is less than unity. When output is defined as *net output* rather than as *gross output,* thus excluding intraindustry transactions, a_{jj} or a_{ii} becomes zero.[16]

As Almon[17] correctly states, all columns of A are less than one (i.e., strict inequality), since each industry has labor and capital input. This point is covered in some detail in the previous chapter. In general, the strict inequality conditions make good sense from an economic standpoint, but are not required mathematically. That is, anticipating a later discussion, the matrix A can be nonsingular and a nonnegative $(I - A)^{-1}$ can exist in the event that at least one column sum of A is equal to unity. We will see that in order for a nonnegative $(I - A)^{-1}$ to exist, or equivalently, in order for the series $I + A + A^2 \ldots + A^r$ to converge to $(I - A)^{-1}$, the *maximal characteristic root* λ_m of A, equal to the *Perron root* ρ (A) of A, must be less than unity in modulus. Although the maximal characteristic root λ_m of A can theoretically be equal to an upper bound defined by the maximum column sum, having at least one (maximum) column sum equal to unity does not by itself preclude the existence of a maximal characteristic root λ_m less than unity in modulus.

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[13] Richard Bellman, *Introduction to Matrix Analysis* (New York: McGraw-Hill Book Company, Inc., 1960), p. 288.

[14] Y.K. Wong, "Some Mathematical Concepts for Linear Economic Models," in Oskar Morgenstern (ed.), *Economic Activity Analysis* (New York: John Wiley and Sons, Inc., 1954), pp. 324-325.

[15] Y.K. Wong and Oskar Morgenstern, "A Study of Linear Economic Systems," *Weltwirtschaftliches Archiv.,* LXXIX (1957 II), 224.

[16] See, for example, W.J. Berger and Edward Saibel, "Power Series Inversion of the Leontief Matrix," *Econometrica,* XXV, 1 (January, 1957), 155.

[17] Clopper Almon, Jr., *Matrix Methods in Economics* (Reading, Mass.: Addison-Wesley Publishing Co., 1967), p. 28.

Apart from this, there seems to be a lack of consistency, as well as general confusion, in the literature on what exactly is meant by a Minkowski-Leontief type matrix. For example, Wong, while he gives a definition in terms of conditions (f), (g), and (h),[18] states elsewhere that "the Minkowski-Leontief matrix I–A is not symmetric."[19] McKenzie[20] more explicitly defines a Minkowski-Leontief matrix as being equivalent to what is generally called the Leontief matrix (I – A), which meets the following conditions: If C = (I – A), then

(i) $c_{ij} \leqslant 0$ for $i \neq j$,

(j) $c_{ij} \geqslant 0$ for $i = j$, and

(k) $\sum_{i=1}^{n} c_{ij} \geqslant 0 \quad (i, j = 1,2,\ldots,n)$.

To avoid further confusion, by a *Minkowski-Leontief type matrix,* we will mean a matrix such as the A matrix of the open-static input-output model, and the name *Leontief matrix* will be reserved for (I – A).

5. The Fundamental Algebraic Eigenproblem:[21] Given a square nonnegative matrix $A = [a_{ij}]$ of order n, the fundamental algebraic eigenproblem is the determination of the numerical values of the scalar λ for which the set of n homogeneous linear equations in n unknowns

(a) $Au = \lambda u$

has a nontrivial solution. This equation can be rewritten as

(b) $Au - \lambda u = 0$,

which is equivalent to

(c) $(A - \lambda I)\, u = 0$.

Here, *o* represents a null vector, and is interchangeably used as a null matrix or as the scalar *zero.*

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[18]Wong, *loc. cit.*

[19]Y.K. Wong, "Inequalities for Minkowski-Leontief Matrices," in Oskar Morgenstern(ed.), *Economic Activity Analysis* (New York: John Wiley and Sons, Inc., 1954), p. 204, footnote 1.

[20]McKenzie, *op. cit.*, 456.

[21]See, for example, J.H. Wilkinson, *The Algebraic Eigenvalue Problem* (Oxford, England: Clarendon Press, 1965), pp. 2-10.

An equation of the form (c) has a nontrivial solution only if the rank of the matrix $(A - \lambda I)$ is less than its order, that is, if the matrix $(A - \lambda I)$ is singular or if its determinant is zero.

(d) $\det(A - \lambda I) = |A - \lambda I| = 0.$

Accordingly, Equation (d), known as the *characteristic equation,* establishes conditions under which Equation (a) is true, namely, values of λ which satisfy (d) are such that (a) is also satisfied. When A is of order n, the characteristic equation is a polynomial in λ of degree n:

(e) $\alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \ldots + \alpha_{n-1} \lambda^{n-1} + (-1)^n \lambda^n = 0.$

Equation (e), called the *characteristic polynomial,* possesses n roots, $\lambda_1, \lambda_2, \ldots, \lambda_n$. In general, these roots may be complex, even if the matrix A is real. Further, there may be roots of any multiplicities up to n. The n roots are called the *eigenvalues, latent roots, characteristic roots (values),* or *proper values* of the matrix A.

For each value of λ, Equation (a) holds true; thus, in general, we should expect to find n vectors $u_1, u_2, \ldots, u_n$ corresponding to each of the n λ's. Accordingly, corresponding to any eigenvalue λ, the set of equations (c) has at least one nontrivial solution u. Such a solution is called an *eigenvector, latent vector, characteristic vector,* or *proper vector* corresponding to that eigenvalue.

Since Equation (a) can alternatively be written as

(f) $\lambda u = Au$

and Equation (c) as

(g) $(\lambda I - A)u = 0,$

then the characteristic equation (d) is equivalent to

(h) $|\lambda I - A| = 0.$

6. Norms (Moduli) of Matrices: The *norm (modulus)* of a finite matrix A with real or complex elements is a finite real-valued function, $\|A\|$, defined as[22]

$$\|A\| = \operatorname*{Max}_{j} \left\{ \sum_{i=1}^{n} \|a_{ij}\| \quad i, j = 1, 2, \ldots, n \right\},$$

[22] Hadley, *op. cit.,* p. 129.

where

$$|a_{ij}| \text{ is the absolute value of } a_{ij},$$

or as

$$\| A \| = \max_{i} \left\{ \sum_{j=1}^{n} |a_{ij}| \quad i, j = 1,2,\ldots,n \right\} .$$

The norm (modulus) of a matrix satisfies the following conditions:[23]

(a) $\| A \| > 0$ unless $A = 0$;

(b) $\| I \| = 1$, independently of the order of the identity matrix I;

(c) $\| kA \| = |k| . \| A \|$ for any scalar k;

(d) $\| A B \| \leqslant \| A \| . \| B \|$;

(e) $\| A + B \| \leqslant \| A \| + \| B \|$;

(f) Let E be a submatrix of the identity matrix I (of order n). Then $\| E \| \leqslant \| I \|$.

(g) $\left| \| A \| - \| B \| \right| \leqslant \| A + B \|$;

(h) If A and B are nonnegative matrices such that $A \geqslant B$, then $\| A \| \geqslant \| B \|$;

(i) $\| -A \| = \| A \|$;

(j) If A consists of a single number s, then $\| s \| = |s|$.

The modulus function is nonnegative and vanishes if and only if A is a zero matrix. A matrix has no numerical value. Thus, the norm of a matrix gives an overall evaluation of the size of the matrix and plays the same role as the modulus in the case of a complex number.[24]

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[23] *Ibid.* Also see Almon, *op. cit.,* pp. 27-30; Varga, *op. cit.,* pp. 7-12; Wilkinson, *op. cit.,* pp. 55-58; and Y. K. Wong, "Quasi-Inverses Associated Minkowski-Leontief Matrices," *Econometrica,* XXII, 3 (July, 1954), 351-352.

[24] The discussion of the norm of matrices by no means stops here. For detailed results and more complete coverage, see Varga, *loc. cit.,* as well as Wilkinson, *loc. cit.*

C. SOME KNOWN PROPERTIES OF POSITIVE AND NONNEGATIVE MATRICES: A GENERAL DISCUSSION

The *technological* coefficients matrix of the basic input-output model, which is a matrix of the Minkowski-Leontief type, is but a special case of the more general class of positive and nonnegative matrices. Many of the important mathematical properties of positive and nonnegative matrices are directly applicable to matrices of the Minkowski-Leontief type. Thus, a general exposition of some of the more important properties of positive and non-negative matrices not only provides a necessary background for the subsequent discussion on matrices of the Minkowski-Leontief type but also comprises an integral part of such discussion.

1. Positive Matrices:

In 1907, Perron[25] established some important properties of the spectrum (i.e., characteristic roots and characteristic vectors) of positive matrices. Perron proved that if the elements of a square matrix $A = [a_{ij}]$ of order n are all positive, then A has the following properties:

(a) A has a positive characteristic root (eigenvalue) $\lambda_m, \lambda_m > 0$, such that

(b) λ_m is a root of the characteristic equation of A, $|I\lambda - A| = 0$, where

(c) λ_m is greatest absolute value (modulus) of all the characteristic roots λ_i $(i = 1,...,n)$[26] and there is but one characteristic root of modulus λ_m;

(d) associated with the characteristic root λ_m is a characteristic vector (eigenvector) u_0 with positive coordinates (i.e., $u_0 > 0$), and λ_m is the only characteristic root of A for which a corresponding characteristic vector with nonnegative components exists.

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[25] Oskar Perron, "Zur Theorie der Matrizes," *Mathematische Annalen*, LXIV (1907), 248-263.

[26] This interpretation (i.e., $\lambda_m > |\lambda_i|$), implying strict inequality, is due to Alfred Brauer. Among his many articles on the subject, see his "On the Theorems of Perron and Frobenius on Non-negative Matrices," in G. Szegö (ed.), *Studies in Mathematical Analysis and Related Topics* (Stanford, Calif.: Stanford University Press, 1962), pp. 48 -55. Gantmacher's interpretation of this Perron result is consistent with that of Brauer [*See Gantmacher, op. cit., p. 64*].

Debreu and Herstein [*op. cit., 598*], on the other hand, give a slightly different interpretation by admitting equality (i.e., $\lambda_m \geqslant |\lambda_i|$). Fan agrees with this interpretation when he states that λ_m is not less than the absolute value of any other characteristic root of A [*See Ky Fan,* "Topological Proofs for Certain Theorems on Matrices with Non-negative Elements," *Monatshefte für Mathematik, LXII (1958), 221*].

In addition to confirming these results, G. Frobenius[27] showed that if

(i) $R_i = \sum_{j=1}^{n} a_{ij}$ represents the sum of the elements of any row ($i = 1,2,\ldots,n$), where $R = \max R_i$ and $r = \min R_i$, and

(ii) $a = \max a_{ii}$ represents the maximum of the elements of the main diagonal, then

(e) the greatest characteristic root λ_m of A lies between the greatest and the smallest row-sum, satisfying the inequalities

$$\text{Min } R_i = r \leqslant \lambda_m \leqslant R = \text{Max } R_i.$$

and

(f) $\lambda_m > a$,

that is, the greatest characteristic root of A is greater than the greatest main diagonal element.[28]

In the literature, the greatest characteristic root, λ_m, of A is termed the *dominant* or *maximal* characteristic root. The *spectral radius* σ (A) of A, defined as the maximum of the moduli $|\lambda_i|$ of all characteristic roots λ_i of A, is thus equal to the maximal characteristic root of A (i.e., $\sigma(A) = \lambda_m$).[29]

2. Nonnegative Matrices:

A positive matrix is a special case of an indecomposable nonnegative matrix. Frobenius[30] extended the results on positive matrices to the class of indecomposable nonnegative matrices. He proved that the results (a) through (f) hold for indecomposable nonnegative matrices, except that the greatest positive root may be equal to the absolute value of some of the other roots (i.e., $\lambda_m = |\lambda_i|$). Frobenius called the matrix imprimitive in such a case, and otherwise primitive. More formally, if $A \geqslant 0$ is an indecomposable nonnegative matrix of order

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[27] G. Frobenius, "Über Matrizen aus positiven Elementen," *Sitzungsberichte der königlich preussischen Akademie der Wissenschaften zu Berlin,* No. 26 (1908), 471-476; and *ibid. (1909), 514-518.*

[28] Also see Alfred Brauer, "The Theorems of Ledermann and Ostrowski on Positive Matrices," *Duke Mathematical Journal,* XXIV (1957), 265.

[29] Varga, *op. cit.,* p. 9.

[30] G. Frobenius, "Über Matrizen aus nicht negativen Elementen," *Sitzungsberichte der königlich preussischen Akademie der Wissenschaften,* 26-27 (1912), 456-477.

n (n $\geqslant$ 2) and if A has k characteristic roots of modulus λ_m, then A is primitive if k = 1 and imprimitive (cyclic) if k > 1. In the latter case, A is called *cyclic of index k.*[31]

Frobenius generalized the basic results on indecomposable nonnegative matrices for the more general class of matrices with nonnegative elements. A summary of the more important results can be given as follows. Let A be a nonnegative matrix. Then there exists a characteristic root ρ (A), called the *Perron root of A,* such that it is nonnegative (i.e., ρ (A) $\geqslant$ 0) but not necessarily simple, it is greater than or equal to the modulus of each other root (i.e., ρ (A) $\geqslant |\lambda_i|$), it is greater than or equal to the smallest row-sum and less than or equal to the greatest row sum (i.e., r $\leqslant \rho$ (A) $\leqslant$ R), it is greater than or equal to the greatest diagonal element (i.e., ρ (A) $\geqslant$ a), and associated with it is a characteristic vector with nonnegative elements. If 0 $\leqslant$ A $\leqslant$ B, then ρ (A) $\leqslant \rho$ (B). Moreover, the *spectral radius* σ (A) of A is defined as the maximum of the moduli $|\lambda_i|$ of all characteristic values of λ_i of A, and according to the Perron-Frobenius results, we have σ (A) = ρ (A) for nonnegative matrices.[32]

Many proofs and extensions of these basic results due to Perron and Frobenius can be found in Alexandroff and Hopf,[33] Wielandt,[34] Debreu and Herstein,[35] Herstein,[36] Fan,[37]

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[31] Varga, *op.cit.,* p. 35. The term "cyclic of index k" for imprimitive matrices was introduced by Romanovsky [See *Romanovsky, loc.cit.*].

[32] See Brauer, "The Theorems of Ledermann and Ostrowski on Positive Matrices," *op.cit.,* 266; Brauer, "On the Characteristic Roots of Non-negative Matrices," in Hans Schneider (ed.), *Recent Advances in Matrix Theory* (Madison and Milwaukee, Wis.: The University of Wisconsin Press, 1964), pp. 30-33.

[33] P. Alexandroff and H. Hopf, *Topologie I* (Berlin: Springer-Verlag, 1935), p. 480.

[34] H. Wielandt, "Unzerlegbare, nicht negative Matrizen," *Mathematische* Zietschrift, LII (1950), 642-648.

[35] Debreu and Herstein, *op.cit.,* 597-607.

[36] I.N. Herstein, "A Note on Primitive Matrices," *American Mathematical Monthly,* LXI (1954), 18-20.

[37] Fan, *op.cit.,* 219-237.

Householder,[38] Holladay and Varga,[39] Pták,[40] Pták and Sedlaček,[41] Gantmacher, [42] Brauer,[43] Varga,[44] Samelson,[45] Ullman,[46] Fiedler and Pták,[47] and Ostrowski.[48] In addition, there have been many attempts at establishing upper and lower bounds on the Perron root ρ (A) of A. If we let $A = [a_{ij}]$ be a square matrix with real or complex elements and

$$R_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad 1 \leqslant i \leqslant n, \tag{1}$$

then all the eigenvalues λ_i of A lie in the interior or on the boundary of at least one of the n disks (circles) with radius z

$$|z - a_{ii}| \leqslant R_i. \tag{2}$$

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[38] A.S. Householder, "On Matrices with Non-negative Elements," *Monatshefte fur Mathematik,* LXII (1958), 238-242.

[39] J.C. Holladay and R.S. Varga, "On Powers of Non-negative Matrices, *Proceedings of the American Mathematical Society,* IX (1958), 631-634.

[40] V. Pták, "On a Combinatorial Theorem and its Application to Non-negative Matrices," *Chekhoslovatskii Matematicheskii Zhurnal,* VIII (1958), 487-495.

[41] V. Pták and J. Sedlaček, "On the Index of Imprimitivity of Nonnegative Matrices," *Chekhoslovatskii Matematicheskii Zhurnal,* VIII (1958), 496-501.

[42] Gantmacher, *op.cit.,* 61-99.

[43] Brauer, "On the Theorems of Perron and Frobenius on Non-negative Matrices," in G. Szegö (ed.), *op.cit.,* pp. 48-55; also "On the Characteristic Roots of Non-negative Matrices," in Hans Schneider (ed.), *op.cit.,* pp. 3-38.

[44] Varga, *op.cit.,* 26-55.

[45] H. Samelson, "On the Perron-Frobenius Theorem," *Michigan Mathematical Journal,* IV (1957), 57-59.

[46] J.L. Ullman, "On a Theorem of Frobenius," *Michigan Mathematical Journal,* I (1952), 189-193.

[47] M. Fiedler and V. Pták, "On Matrices with Non-positive Off-diagonal Elements and Positive Principal Minors," *Chekhoslovatskii Matematicheskii Zhurnal,* XII (1962), 382-400.

[48] A.M. Ostrowski, "On Positive Matrices," *Mathematische Annalen,* CL (1963), 276-284.

This is a well known result, due to a series of articles by Rohrback,[49] Gerschgorin,[50] Specht,[51] Barankin,[52] Taussky,[53] Bodewig,[54] and Brauer.[55]

If we let a_{ii} and a_{jj} represent any two elements of the main diagonal of A and R_i is as defined earlier, then each characteristic root λ_i of A lies in the interior or on the boundary of at least one of the $\binom{n}{2}$ ovals of Cassini, as proved by Brauer:[56]

$$(3) \qquad |z - a_{ii}|\ |z - a_{jj}| \leqslant R_i\ R_{jj} \quad (i, j = 1,2,\ldots,n; i \neq j).$$

A single oval has to be considered, since it contains all the other ovals. Hence, every point of the oval represented by (3) lies in at least one of the disks $|z - a_{ii}| \leqslant R_i$ and $|z - a_{jj}| \leqslant R_j$ (for $j \neq i$). Consequently, the union of the disks (2) contains the union of the ovals.[57]

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[49] H. Rohrbach, "Bemerkungen zu einem Determinantensatz von Minkowsky," Jahresbericht *den Deutschen Mathematiker Vereinigung,* XL (1931), 49-53.

[50] S. Gerschgorin, "Über die Abgrenzung der Eigenwerte einer Matrix," *Bulletin de l' Académie des Sciences de l' URSS, Classe Mathematique,* 7th serié (1931), 749-754.

[51] W. Specht, "Zur Theorie der Algebraischen Gleichungen," *Jahresbericht der Deutschen Mathematiker Vereinigung,* XLVIII (1938), 142-145.

[52] E.W. Barankin, "Bounds for the Characteristic Roots of a Matrix," *Bulletin of the American Mathematical Society,* LI (1945), 767-770.

[53] Olga Taussky, "Bounds for Characteristic Roots of Matrices," *Duke Mathematical Journal,* XV (1948), 1043-1044; also "Bounds for Characteristic Roots of Matrices II," *Journal of Research of the National Bureau of Standards,* XLVI, 2, Research Paper 2184 (February, 1951), 124-125.

[54] E. Bodewig, *Matrix Calculus* (Revised Edition; New York: Interscience Publishers, Inc., 1959).

[55] Alfred Brauer, "Limits for the Characteristic Roots of a Matrix," *Duke Mathematical Journal,* XIII (1946), 387-395.

[56] Brauer, "On the Characteristic Roots of Non-negative Matrices," in Hans Schneider (ed.), *op. cit.,* pp. 5-7.

[57] *Ibid.,* p. 7.

Much sharper results and generalizations can be obtained on bounds for characteristic roots, as given in a series of papers by Ledermann,[58] Ostrowski,[59] Ostrowski and Schneider,[60] and Crabtree.[61] Further, various properties of the maximal characteristic root are given in two recent articles by Brualdi[62] and Yamamoto.[63] In another recent article, Brauer[64] provides a useful method for computing the maximal root of a nonnegative matrix.

In fact, computing characteristic roots and characteristic vectors on a digital computer, determining their accuracy, and estimating the effect of the various errors inherent in the formulation and solution of the algebraic eigenvalue problem have been a subject of major concern in numerical analysis in recent years.[65]

D. EXTENSIONS TO MATRICES OF THE MINKOWSKI-LEONTIEF TYPE

As indicated earlier, the preceding discussion on positive and nonnegative matrices is both an integral part of and provides a necessary background for the more *focused* discussion here on matrices of the Minkowski-Leontief type. To avoid repetition, emphasis here is placed on certain important extensions of the earlier results, and more significantly, on some new results that are more specifically related to matrices of the Minkowski-Leontief type (i.e., *technological* coefficients matrices in input-output analysis).

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[58] Walter Ledermann, "Bounds for the Greatest Latent Roots of a Positive Matrix," *The Journal of the London Mathematical Society,* XXV (1950), 265-268.

[59] A.M. Ostrowski, "Bounds for the Greatest Latent Root of a Positive Matrix," *The Journal of the London Mathematical Society,* XXVII (1952), 253-256.

[60] A.M. Ostrowski and Hans Schneider, "Bounds for the Maximal Characteristic Root of a Nonnegative Irreducible Matrix," *Duke Mathematical Journal,* XXVII (1960), 547-553.

[61] Douglas E. Crabtree, "On the Characteristic Roots of Matrices," *Proceedings of the American Mathematical Society,* XVI (1965), 1410-1413.

[62] Richard A. Brualdi, "On the Permanent and Maximal Characteristic Root of a Nonnegative Matrix," *Proceedings of the American Mathematical Society,* XVII, 6 (December, 1966), 1413-1416.

[63] Tetsuro Yamamoto, "On the Extreme Values of the Roots of Matrices," *Journal of the Mathematical Society of Japan,* XIX, 2 (April, 1967), 173-178.

[64] Brauer, *op. cit.,* 564-569 ["A Method for the Computation of the Greatest Root of a Nonnegative Matrix"].

[65] Wilkinson, *op. cit.,*This is an excellent source on the subject for the interested reader.

The discussion here is organized around five important topics: (1) characteristic roots, (2) indecomposability, (3) decomposability and triangulation, (4) complete decomposability, and (5) dominant diagonality. There are, to be sure, some topics that are omitted from the discussion here, such as those relating to the determinants of Minkowski-Leontief matrices, convergence properties of the matrix A (i.e., $\lim_{p \to \infty} A^p = 0$ if and only if the maximal characteristic root of A is less than unity, that is $\lambda_m < 1$), and a few other subjects, but these will be more appropriately covered later in discussing the properties of M-matrices and the Leontief matrix.

1. Characteristic Roots

The characteristic roots of the transpose of a matrix, A′, are the same as those of A, while the characteristic vectors are, in general, different. Because of the similarity property with respect to characteristic roots, all results on characteristic roots expressed in terms of the row elements of a matrix can be equally well expressed in terms of column elements. This is quite important, since by this fact alone we can directly translate the results discussed earlier into the properties of Minkowski-Leontief matrices.

To start with, in the case of matrices of the Minkowski-Leontief type, the maximal characteristic root now has upper bound that is equal to the maximum column sum, which, as we already know, is less than unity.

If $A = [a_{ij}]$ is a nonnegative, decomposable or indecomposable matrix, none of whose column sums is greater than one, and at least one of whose column sums is less than one, then a sufficient condition that all the characteristic roots λ_i of A be less than one in modulus is that each diagonal submatrix $A_{11}, A_{22}, \ldots, A_{nn}$ have at least one column sum less than unity, as proved by Solow.[66] About a decade later, Fisher[67] gave an alternative proof of a rather more generalized version of Solow's theorem, in which he has shown the exact relationship between the largest root and the column sums. His proof is rather simple,

[66] Robert Solow, "On the Structure of Linear Models," *Econometrica,* XX, 1 (January, 1952), 36-38.

[67] Franklin M. Fisher, "An Alternate Proof and Extension of Solow's Theorem on Nonnegative Square Matrices," *Econometrica,* XXX, 2 (April, 1962), 349-350.

but his presentation is unnecessarily terse and unclear. If we let $A \geqslant 0$ be an nxn matrix with largest characteristic root λ_m, which is real and positive, then we can write

(4) $$Au_0 = \lambda_m u_0$$

where u_0 stands for the particular characteristic vector associated with the maximal characteristic root. Filling in, along the way, many gaps that exist in Fisher's original presentation and by modifying it, we can proceed as follows:

Since

(5) $$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \cdot & \cdot\ \cdot & & \cdot \\ \cdot & \cdot\ \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot\ \cdot \\ \cdot & \cdot & & \cdot\ \cdot \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{pmatrix} \bar{u}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \bar{u}_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_{ij}\ \bar{u}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \sum_{j=1}^{n} a_{nj}\ \bar{u}_n \end{pmatrix} \quad i, j = 1, 2, \dots, n,$$

where u_0 is an nx1 column vector, a particular component of which is denoted by $\bar{u}_i$. Since the right hand side of (5) can be expressed more compactly as an nx1 vector

$\left(\sum_{j=1}^{n} a_{ij}\ \bar{u}_i \right)$, then we have $Au_0 = \left(\sum_{j=1}^{n} a_{ij}\ \bar{u}_i \right)$.

Consequently, we can re-write (4) as

(6) $$\left(\sum_{j=1}^{n} a_{ij}\ \bar{u}_i \right) = \lambda_m u_0,$$

or as

(7) $$\left(\sum_{j=1}^{n} a_{ij}\ \bar{u}_i \right) = \lambda_m (\bar{u}_i), \quad \text{since } u_0 = (\bar{u}_i).$$

Summing over i (rows) we now have

(8) $$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\ \bar{u}_i = \sum_{i=1}^{n} \lambda_m\ (\bar{u}_i),$$

where

(9) $$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\ \bar{u}_i = \sum_{j=1}^{n} (\bar{u}_i \sum_{i=1}^{n} a_{ij})\ ;$$

(10) $$\sum_{i=1}^{n} \lambda_m\ (\bar{u}_i) = \lambda_m \left[\sum_{i=1}^{n} (\bar{u}_i) \right].$$

Since

(11) $$\text{Min} \sum_{i=1}^{n} a_{ij} \leqslant \lambda_m = \rho\,(A) \leqslant \text{Max} \sum_{i=1}^{n} a_{ij},$$

the column sum $\sum_{i=1}^{n} a_{ij}$ in Equations (6) through (9) need not be restricted to the maximal column sum. However, in the special case where the maximal characteristic root is in fact equal to the maximal column sum, we have, re-writing Equations (4) through (7),

(12) $$\sum_{j=1}^{n} \bar{a}_{ij}\ \bar{u}_i = \lambda_m\ (\bar{u}_i)$$

where $\bar{a}_{ij}$ denote the elements of the maximal column sum. In such a case, Equation (8) reduces to

(13) $$\lambda_m \ \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \bar{a}_{ij}\ \bar{u}_i}{\sum_{i=1}^{n} (\bar{u}_i)}$$

where λ_m is the nonnegatively weighted average of the maximal column sum. The weights are the nonnegative components of the characteristic vector associated with the maximal characteristic root, λ_m, and are applied to the elements of the *maximal sum* column.

2. Indecomposability

From an economic standpoint, the condition of indecomposability means that "... no industry can produce *anything* without derived demand being felt in all industries, so that all industries must produce *something.*"[68]

As indicated earlier, if the *technical* coefficients or *technology* matrix, A, of the basic input-output model is a positive matrix ($A > 0$), we can readily say that it is indecomposable. Let us now assume that A is nonnegative (i.e., $A \geqslant 0$). If, in addition, we assume that it is indecomposable, the $A^p > 0$ for some positive integer p if and only if A is primitive.[69] In other words, the series $I + A + A^2 + ... + A^p$, converging to the inverse of the Leontief matrix $(I - A)^{-1}$, will yield a strictly positive inverse matrix if and only if A is primitive. If A is primitive, then A^p is also primitive for all positive integers p,[70] which is another way of saying that the Leontief inverse $(I - A)^{-1}$ is also primitive.

Now, let us assume only that A is nonnegative. Further, let us now express the basic input-output formulation by the difference equation system

$$(14) \qquad I\,X(t+1) - A\,X(t) = Y,$$

obtained by putting $X(t+1) = X(t) = X$, a column of unknown constants. The solution of the set of linear equations $(I - A)\,X = Y$ can easily be recognized as the statical solution of this difference equation system. By iteration, its solution is seen to be expressible as

$$(15) \qquad X(t) = A^p\,X(0) + (I + A + A^2 + ... + A^p)\,Y.$$

As we shall see, this solution will converge to the solution of $X = (I - A)^{-1}$ if and only if the maximal characteristic root λ_m is less than one in modulus, in which case the infinite multiplier series $I + A + A^2 + ... + A^p$ converges to the matrix $(I - A)^{-1}$ and A^p tends to

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[68] Solow, *op. cit.,* 39.

[69] Varga, *op.cit.,* p. 41. Alternatively, $A^p > 0$ if and only if A is indecomposable with positive diagonal elements, $a_{ii} > 0$ [*Ibid.*]. Thus, Solow is slightly in error when he states "...If a is indecomposable, the sum $I + a + a^2 + ... + a^n$ is a matrix all of whose elements are strictly positive and hence the same is true of the infinite matrix series converging to $(I-A)^{-1}$" [See Solow, *op.cit.,* 40]. The additional restriction that he omits is that the diagonal elements of A be positive.

[70] *Ibid.* [*Varga*] , p. 40.

the null matrix. If this is so, then clearly the components of the solution vector must be nonnegative, since the powers of A will be nonnegative and Y is nonnegative by definition. Evidently, then, the conditions for the nonnegativity of $(I - A)^{-1}$ and for the stability of the difference equation system (14) are closely related,[71] where we only require that in order for the difference equation system to be considered stable all characteristic roots λ_i of A must be less than unity in modulus.

Then, if A is primitive, the dominant motion of the dynamic difference equation system (14)

$$I\,X(t+1) - A\,X(t) = Y$$

is monotonic; that is, for general initial conditions and sufficiently large p, any oscillatory components will be negligible compared with the motion due to the maximal characteristic root λ_m. In terms of the iteration (15), eventually it will be true that $X(t+1) - X(t)$ will no longer change sign if A is primitive. On the other hand, if A is imprimitive, then there will be an oscillatory mode due to a negative or complex root which will damp no faster than the monotonic motion.[72]

3. Decomposability and Triangulation

In order for a nonnegative nxn matrix A to be decomposable, there must exist an nxn permutation matrix P such that we obtain

$$PAP' = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{16}$$

where P′ is the transpose of P, A_{11} is an rxr submatrix and A_{22} is an $(n-r)x(n-r)$ submatrix, with $1 \leqslant r \leqslant n$. We now ask again if A_{11} and A_{22} are indecomposable, and if not,

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[71] Solow, *op. cit.*, 31

[72] *Ibid.*, 40.

we decompose them in the manner we initially decomposed the matrix A. Thus, there exists an nxn permutation matrix P such that

$$(17) \qquad PAP' = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ 0 & R_{22} & \cdots & R_{2m} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & R_{mm} \end{bmatrix}$$

where each submatrix R_{ij}, $1 \leqslant j \leqslant m$, is either indecomposable or a lxl null matrix.

In a triangulated (triangularized) Minkowski-Leontief matrix having the form (17) the submatrices along the principal diagonal, R_{ii} or R_{jj}, must be indecomposable and non-null in order for it (i.e., the resulting triangular matrix) to be nonsingular. This is so, since a triangular matrix is nonsingular if and only if all the diagonal elements are nonzero.[73] This condition derives from the fact that the determinant of a triangular matrix is equal to the product of the elements on the principal diagonal.[74] Thus, the condition of indecomposability for the submatrices along the principal diagonal precludes the possibility of any diagonal submatrix containing a row or column of zeroes.[75]

The problem of triangulation has been discussed quite extensively in the input-output literature.[76] The remaining comments here, therefore, will be limited to only a few, general observations.

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[73] Ben Noble, *Applied Linear Algebra* (Preliminary Edition; Englewood Cliffs, N.J.: Prentice Hall, Inc., 1966), p. 102.

[74] Louis G. Kelly, *Handbook of Numerical Methods and Applications* (Reading, Mass.: Addison-Wesley Publishing Co., 1967), p. 111.

[75] D.V.T. Bear, D. Jorgenson, and H.M. Wagner, "Elementary Proofs of Propositions on the Leontief-Minkowski Matrices," *Metroeconomica,* XIV, 1-2-3 (April-August-December, 1962), 59.

[76] For a sample, see H. Aujac, "La hiérarchie des industries dans un tableau des échanges interindustriels," *Revue économique,* XI, 2 (March, 1960), 169-238; D. Masson, "Methode de triangulation du tableau *européen* des échanges interindustriels," *Ibid.,* 239-265; Ernst Helmstädter, "Die geordnete Input-Output struktur," *Jahrbücker für Nationalökonomie und Statistik,* CLXXIV, 4-5-6 (1962), 322-361.

The triangulated form (17) is called the *normal form of a decomposable matrix A*.[77] Although the normal form is shown here as being *upper triangular*, it can alternatively be *lower triangular*, with zeroes everywhere above the principal diagonal.

The inverse of an upper (lower) triangular matrix is also an upper (lower) triangular matrix.[78] Further, referring again back to (17), the characteristic roots of the Minkowski-Leontief matrix A are the same as the characteristic roots of the square submatrices R_{ii} along the principal diagonal.[79]

The economic significance of the triangular form is that it enables a clear and orderly view of the hierarchy of intersectoral interdependence patterns existing in an economic system. In practice, a triangulated *transactions* or Minkowski-Leontief matrix would not have the perfect symmetry that the mathematical discussion here would suggest. One would ordinarily observe a scattering of non-zero entries above or below the principal diagonal, depending on one or the other triangular form sought, where one would expect to see all zeroes.

Further, the triangular form is useful in solving the system. If we have an upper triangular *technological* matrix, then the Leontief matrix will have positive elements on the principal diagonal, negative or zero elements above and zero elements below the principal diagonal:

$$(18) \qquad \begin{bmatrix} (1-a_{11}) & -a_{12} & \cdots & -a_{1n} \\ 0 & +(1-a_{22}) & \cdots & -a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & (1-a_{nn}) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix}.$$

Then, the solution of the nth equation is trivial:

$$(1-a_{nn})\,x_n = y_n, \text{ or } x_n = y_n/(1-a_{nn}).$$

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[77] Varga, *op. cit.*, p. 46. Also see Gantmacher, *op. cit.*, p. 90.

[78] Noble, *loc. cit.*

[79] Varga, *loc. cit.*

In the (n − 1)th equation we then have

$$(-a_{n-1,n-1})\ (x_{n-1}) + (-a_{n-1,n})\ (x_n) = y_{n-1},$$

into which we can substitute the already found value of x_n and solve for x_{n-1}. Proceeding in this way, from the *bottom up,* we can easily solve the entire system.

4. Complete Decomposability

We suppose now that a Minkowski-Leontief matrix A is completely decomposable. This would require the existence of an nxn permutation matrix P such that we obtain

$$\text{(19)} \qquad PAP' = \begin{bmatrix} R_{11} & 0 & \dots & 0 \\ 0 & R_{22} & \dots & 0 \\ . & . & . & . \\ 0 & 0 & & R_{mm} \end{bmatrix}$$

where each square submatrix R_{ii} along the principal diagonal is indecomposable.[80] In such a system, each group or *cluster* of sectors defined by R_{ii} would be independent of the other groups. We would thus have m different interindustry trading systems within an economy. The implication of this is that given an exogenous change in final demand for the products of a sector in say, block R_{11}, there will result no derived demand for the products of industries not in block R_{11}, since the industries in block R_{11} use only the products of one another. By contrast, in a normal indecomposable matrix, a change in final demand will eventually, after at most n − 1 *rounds*, create repercussions potentially on every other sector.

In the complete decomposability case, each *cluster* is independent of those appearing before or after itself; the simultaneity of the entire problem has now disappeared. Here, each cluster affects the economic system, but there is no feedback, or reverse effect, since exogenous changes in final demand for the products of industries in a given *cluster* will have effects that are totally contained within that particular cluster.

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[80]Solow, *op. cit.,* 34.

In practice, there exist such *substantially independent* groups of industries, although not in as neat and orderly a fashion as would be suggested by the complete decomposability case. A discussion of this can be found, for example, in an article by Ghosh.[81]

5. Dominant Diagonality

A nonnegative Minkowski-Leontief matrix is said to have a dominant diagonal if $|a_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}|$ for each j. This definition is due to McKenzie.[82] In other words, the dominant diagonality condition states that the absolute value of each diagonal element must be greater than the sum of the absolute values of all other elements in the same column. This definition can be generalized slightly by introducing numbers $d_j > 0$ such that

$$(20) \qquad d_j \, |a_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^{n} d_i \, |a_{ij}| \qquad \text{for } j = 1,2, \ldots, n.$$

McKenzie defines an even more general type of diagonal dominance, which he terms *quasi-dominance.* An nxn matrix A has a *quasi-dominant diagonal (q.d.d.)* if, first, there exist $d_j > 0$ such that

$$(21) \qquad d_j \, |a_{jj}| \geqslant \sum_{\substack{i=1 \\ i \neq j}}^{n} d_i \, |a_{ij}| \qquad (j = 1,2, \ldots, n),$$

and, secondly, when $a_{ij} = 0$ (given $j \in J$ and $I \notin J$ for some set of indices J), the strict inequality holds for some $j \in J$.[83]

Omitting proofs, we can now present some of the more important properties of quasi-dominant matrices, given by McKenzie, which are particularly relevant for Minkowski-Leontief matrices:

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[81] A. Ghosh, "Input-Output Analysis with Substantially Independent Groups of Industries," *Econometrica,* XXVIII, 1 (1960), 88-96.

[82] Lionel McKenzie, "Matrices with Dominant Diagonals in Economic Theory," in Kenneth J. Arrow, Samuel Karlin, and Patrick Suppes (eds.), *Mathematical Methods in the Social Sciences* (Stanford, Calif.: Stanford University Press, 1960), p. 47.

[83] *Ibid.,* 48.

(a) If a matrix A has a q.d.d., then A is nonsingular;

(b) If a matrix A has a q.d.d. that is negative, all its characteristic roots have negative real parts;

(c) If λ_i is a characteristic root of a matrix A, then

$$\lambda_i \leqslant m = \operatorname{Max} \sum_{i=1}^{n} |a_{ij}| \quad \text{for } j = 1,2,\ldots,n.$$

(d) Let C be a square matrix with $c_{ii} > 0$ for all i and $c_{ij} \leqslant 0$ for $i = j$. Then a necessary and sufficient condition for $C X = Y$ to have a unique solution $X \geqslant 0$ for every $Y \geqslant 0$ is that C have q.d.d.

E. M-MATRICES AND THE PROPERTIES OF THE LEONTIEF MATRIX

A real nxn ($n \geqslant 2$) matrix $C = (c_{ij})$ with $c_{ij} \leqslant 0$ ($i \neq j$) and $c_{ii} > 0$ is called an M-matrix if it has the form $\omega I - A$, where A is a nonnegative matrix and I is the identity matrix, and if it fulfills the following equivalent necessary and sufficient conditions:

(a) $\omega > \rho(A)$.

(b) The leading principal minors of $\omega I - A$ are positive.

(c) All principal minors of $\omega I - A$ are positive.

(d) $\omega I - A$ is nonsingular. In the inverse matrix $(\omega I - A)^{-1} = (\alpha_{ij})$, all elements α_{ij} are nonnegative. Further, if A is indecomposable, then $(\omega I - A)^{-1} > 0$ provided that $\omega > \rho(A)$.

(e) All characteristic roots λ_i of A are positive.

(f) If there exists a vector $X \geqslant 0$, then $(\omega I - A) X \geqslant 0$.

(g) If there exists a vector $X > 0$, then $(\omega I - A) X > 0$.

The Leontief matrix $(I - A)$ is but a special case of M-matrices, where $\omega = 1$. Thus, the properties of M-matrices have direct applicability to the Leontief matrix. Some of these properties have been discovered independently by mathematical economists, but many of them have been investigated by mathematicians over the years in a long series of articles published in numerous mathematical journals that normally have not been readily accessible to economists.

M-matrices, or matrices with non-positive off-diagonal elements, were first introduced and studied by Ostrowski[84] under the names of *eigentliche M-Determinanten* and *eigentliche M-Matrizen.* They have since been investigated by many authors, notably by Fan,[85] Householder,[86] Fan and Householder,[87] Fiedler and Pták,[88] Carlson,[89] and Crabtree.[90]

Much earlier (1887), Stieltjes[91] had proved that if C is a real symmetric and positive definite nxn matrix with all its off-diagonal entries negative, then $C^{-1} > 0$. Later (1912) Frobenius[92] proved the stronger result that if $A > 0$ is an nxn matrix, and ω is a real number with $\omega > \rho(A)$, then the matrix $\omega I - A$ is nonsingular, and $(\omega I - A)^{-1} > 0$. Thus, due to Frobenius, the relationship (a) $\leftrightarrow$ (d) has been well known, as well as the relationship (a) $\rightarrow$ (c). Ostrowski[93] proved (c) $\rightarrow$ (d), and (c) $\rightarrow$ (g). The relationship (c) $\rightarrow$ (d) is also given by Goheen[94] and

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[84] A . M. Ostrowski, "Über die Determinanten mit Überwiegender Hauptdiagonale," *Commentarii Mathematici Helvetici,* X (1937), 69-96; and "Determinanten mit Überwiegender Hauptdiagonale and die absolute Konvergenz von linearen Iterationsprozessen," *Commentarii Mathematici Helvetici,* XXX (1956), 175-210.

[85] Ky Fan, "Topological Proofs for Certain Theorems on Matrices with Nonnegative Elements," *Monatshefte für Mathematik,* LXII (1958), 219-242; also "Note on M-Matrices," *Quarterly Journal of Mathematics,* XI (1960), 43-49.

[86] A. S. Householder, "On Matrices with Non-Negative Elements," *Monatshefte für Mathematik,* LXII (1958), 238-242.

[87] Ky Fan and A. S. Householder, "A Note Concerning Positive Matrices and M-Matrices," *Monatshefte für Mathematik,* LXIII (1959), 265-270.

[88] Fiedler and Pták, *loc. cit.*

[89] David Carlson, "A Note on M-Matrix Equations," *Journal of the Society for Industrial and Applied Mathematics,* XI, 4 (December, 1963), pp. 1027-1033.

[90] Douglas E. Crabtree, "Applications of M-Matrices to Non-Negative Matrices," *Duke Mathematical Journal,* XXXIII (1966), 197-207.

[91] T.J. Stieltjes,"Surles racines de l'equation $X_n=0$," *Acta Mathematica,* IX (1887), 385-400. A real matrix A is symmetric if $A = A'$, where A' is the transpose of A. Further, a matrix A is positive definite if the corresponding quadratic form $X'AX > 0$ for all $X \neq 0$, where X is a column vector and X' is its transpose. See Noble, *op. cit.*, p. 10, 367.

[92] Frobenius, *loc. cit.,* ["Über Matrizen aus nicht negativen Elementen."]

[93] Ostrowski, *loc. cit.,* ["Über die Determinanten mit Überwiegender Hauptdiagonale."]

[94] H. E. Goheen, "On a Lemma of Stieltjes on Matrices," *American Mathematical Monthly,* LVI (1949) 328-329.

Egervary.[95] The equivalence of (b) and (c) is given by Kotelyanskii,[96] whose proofs can be found in Gantmacher.[97]

In addition, Wong[98] showed that if $\| A \|$ is defined to be the maximum of the column sums of A, the condition $\lim_{p\to\infty} \| A^p \|^{1/p} < t$, $t > 0$, is necessary and sufficient for the existence of the inverse $(tI - A)^{-1}$, which is necessarily nonnegative and is positive if A is indecomposable. Thus, he in essence proved the relationship (a)→ (d), by making use of the Cauchy-Hadamard condition $\lim_{p\to\infty} \|A^p\|^{1/p} < 1$. The relationship $\lim_{p\to\infty} \|A^p\|^{1/p} = \rho(A)$, where $\rho(A)$ is the *Perron root* (i.e., maximal characteristic root) of A, is well known as a special case in the theory of Banach algebra[99] and various proofs of it have been given by Gautschi[100] and more recently by Yamamoto.[101]

The topic of M-matrices continued to receive attention among mathematicians. For example, in an exhaustive article, Fan[102] gave topological proofs of the relationships (a) → (d), (a) → (c), (c) → (b), and (b) → (a). In the subsequent and equally comprehensive article, Fiedler and Pták[103] presented a systematic treatment of many of the earlier known results and provided proofs of the chain of relationships given here. These two, in fact, stand as landmark articles on the subject.

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[95] E. Egervary, "On a Lemma of Stieltjes on Matrices," *Acta Scient. Math. Szeged,* XV (1953-1954) 99-103.

[96] D. M. Kotelyanskii, "Some Properties of Matrices with Positive Elements," *Matematicheskii Sbornik,* XXXI 73 (1952), 497-506.

[97] Gantmacher, *op. cit.,* 86-89.

[98] Y. K. Wong, "On Nonnegative-valued Matrices," *Proceedings of the National Academy of Sciences,* XL (1954), 121-124; also see the author's "Inequalities for Minkowski-Leontief Matrices," *op. cit.,* p. 202, pp. 220-222.

[99] L. H. Loomis, *An Introduction to Abstract Harmonic Analysis* (New York: D. van Nostrand Co., 1953).

[100] W. Gautschi, "The Asymptotic Behavior of Powers of Matrices," I and II, *Duke Mathematical Journal,* XX (1953), 127-140 and 375-379.

[101] Yamamoto, *op. cit.,* 174.

[102] Fan, *loc. cit.* ["Topological Proofs for Certain Theorems on Matrices with Non-negative Elements"].

[103] Fiedler and Pták, *loc. cit.*

Very recently, using the fact that M-matrices have nonnegative inverses, Crabtree[104] further improved the upper and lower bounds on the characteristic roots of nonnegative suqare matrices. As before, let R_i represent the sum of the elements of any row $(i = 1,2,...,n)$, where $r = \text{Min } R_i$ and $R = \text{Max } R_i$. We know already that $\text{Min } R_i = r \leqslant \rho(A) \leqslant R = \text{Max } R_i$. Further let A be a nonnegative square matrix of order n, let $\omega > \rho(A)$, and let $C = \omega I - A$. Then, Crabtree showed that

$$(22) \qquad \omega - \frac{1}{r(C^{-1})} \leqslant \rho(A) \leqslant \omega - \frac{1}{R(C^{-1})}$$

where $r(C^{-1})$ and $R(C^{-1})$ respectively stand for the minimum and maximum row-sum in $(\omega I - A)^{-1}$. He subsequently showed that if $C = [c_{ij}] = (\omega I - A)$ is an M-matrix of order n, then

$$(23) \qquad r(C) \leqslant \frac{1}{R(C^{-1})}$$

and

$$(24) \qquad R(C) \geqslant \frac{1}{r(C^{-1})} .$$

Finally, as a corollary, he obtained

$$(25) \qquad \omega - \frac{1}{R(C^{-1})} \leqslant R(A)$$

and

$$(26) \qquad \omega - \frac{1}{r(C^{-1})} \geqslant r(A) .$$

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[104] Crabtree, *loc. cit.*

Among economists, perhaps the best known result over the years has been due to Hawkins and Simon,[105] who proved the relationships (c) → (d) and (c) → (g). The condition that all principal minors of (I − A) must be positive in order for the solution vector $X = (x_j)$ satisfying the system $X = (I - A)^{-1}\ Y$ to be positive came to be known as the *Hawkins-Simon condition(s).* Before Hawkins and Simon, Mosak[106] had proved, in a different connection, that the inverse of the Leontief matrix is positive if and only if its principal minors are positive. The same result was suggested by Georgescu-Roegen,[107] in his Theorem 7. These results, based on the positivity of the principal minors of (I − A), represented logical extensions of previously established results on the determinants of matrices. Bray,[108] for example, had shown considerably earlier that if any principal minor vanishes then the determinant vanishes.

Wong[109] and Woodbury[110] gave further proofs of the positivity of all principal minors of Minkowski-Leontief matrices, Wong utilizing certain results on the non-increasing property of the determinants of (I − A) and Woodbury showing that the principal minors of (I − A) are nonnegative if the real matrix $A = [a_{ij}]$ satisfies $\sum_{i=1}^{n} |a_{ij}| = s_j \leqslant 1$, and they are positive if A satisfies $\sum_{i=1}^{n} |a_{ij}| = s_j < 1$, for $j = 1,2,\ldots,n$.

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[105] David Hawkins and Herbert A. Simon, "Note: Some Conditions of Macroeconomic Stability," *Econometrica,* XVII, 3 and 4 (July-October, 1949), 245-248.

[106] S. L. Mosak, *General Equilibrium Theory in Internatonal Trade* (Bloomington, Indiana: Principia Press, 1944), pp. 49-51. In retrospect, he might have additionally stipulated that the matrix A be indecomposable.

[107] Nicholas Georgescu-Roegen, "Some Properties of a Generalized Leontief Model," in Tjalling C. Koopmans (ed.), *Activity Analysis of Production and Allocation,* Proceedings of a Conference, Cowles Commission for Research in Economics (New York: John Wiley and Sons, Inc., 1951), p. 169.

[108] Hubert E. Bray, "Rates of Exchange" *American Mathematical Monthly,* XXIX (1922), 365-371.

[109] Wong, *op. cit.* 257-263, ["Inequalities for Minkowski-Leontief Matrices"].

[110] Max A. Woodbury, "Properties of Leontief-type Input-Output Matrices," in Oskar Morgenstern (ed.), *Economic Activity Analysis* (New York: John Wiley and Sons, Inc., 1954), 344-348.

The results, plus many more obtained by Arrow,[111] Chipman,[112] Goodwin,[113] Metzler,[114] Morishima,[115] and Solow[116] are generally contained in the relationships (a) → (c), (a) → (d), (a) → (f), (a) → (g), which were proved in a landmark article by Debreu and Herstein.[117]

Lastly, utilizing the fundamental mathematical result, due to Hadamard,[118] that a matrix with a dominant diagonal is nonsingular, McKenzie[119] later established the equivalence of the Hawkins-Simon condition(s) to the condition that in order for $(I - A)X = Y$ to have a unique solution $X \geqslant 0$ for every $Y \geqslant 0$, $(I - A)$ must have a *q.d.d.* (quasi-dominant diagonal).

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[111] K. J. Arrow, "Alternative Proof of the Substitution Theorem for Leontief Models in the General Case," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation* (New York: John Wiley and Sons, Inc., 1951), pp. 155-164.

[112] J. S. Chipman, "The Multi-Sector Multiplier," *Econometrica,* XVIII (October, 1950), 355-374; also *The Theory of Inter-Sectoral Money Flows and Income Formation* (Baltimore: The John Hopkins Press, 1951), Part III.

[113] R. M. Goodwin, "Does the Matrix Multiplier Oscillate," *The Economic Journal,* LX (December, 1950), 764-770.

[114] L. A. Metzler, "Stability of Multiple Markets: The Hicks Conditions," *Econometrica,* XIII (October, 1945), 277-292; "A Multiple-Region Theory of Income and Trade," *Econometrica,* XVIII (October, 1950), 329-354; "Taxes and Subsidies in Leontief's Input-Output Model," *The Quarterly Journal of Economics,* LXV (August, 1951), 433-438.

[115] M. Morishima, "On the Laws of Change of the Price System in an Economy which Contains Complementary Commodities," *Osaka Economic Papers,* I (1952), 101-113.

[116] Solow, *op. cit.,* 31-38.

[117] Debreu and Herstein, *loc. cit.*

[118] Any determinant of order n with elements a_{ij} is different from zero if the elements satisfy the n inequalities

$$\sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}| < |a_{ii}| \qquad (i,j = 1,2,\ldots,n).$$

See J. Hadamard, *Leçons sur la propagation des ondes* (Paris: Herman, 1903) p. 13. This theorem is also cited in Berger and Saibel, *op. cit.,*155-156.

[119] McKenzie, *op.cit.,* 50 ["Matrices with Dominant Diagonals and Economic Theory"].

F. EQUIVALENT NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A UNIQUE NONNEGATIVE SOLUTION TO THE BASIC MODEL

It is clear from the immediately preceding discussion that there are a number of equivalent necessary and sufficient conditions whose fulfillment guarantees the existence of a unique nonnegative solution to the basic input-output model, $X = (I - A)^{-1} Y$, corresponding to a nonnegative final demand vector $Y \geqslant 0$. These equivalent necessary and sufficient conditions can be conveniently separated into four distinct groups, which may be termed *M-matrices, convergence, Minkowski-Leontief,* and *dominant diagonality* conditions. These terms by themselves may be somewhat misleading; thus, further clarification will probably be helpful.

First, the *M-matrices* conditions, refer to the set of conditions derived from the fact that a Leontief matrix $(I - A)$ is an M-matrix $(\omega I - A)$, such that in the Leontief case $\omega = 1$. Since $\omega = 1 > \rho(A)$, where $\rho(A)$ is the *Perron root* or maximal characteristic root of the Minkowski-Leontief matrix A, then all of the equivalent necessary and sufficient conditions just discussed in relation to M-matrices hold for the Leontief matrix. These are also equivalent conditions for the existence of a unique solution to the basic input-output model.

Secondly, the *convergence* condition derives from the fact that $\rho(A) < 1$, which is a necessary and sufficient condition for the infinite multiplier series $I + A + A^2 + \ldots + A^p + \ldots$ to converge to $(I - A)^{-1}$, which is nonnegative if $A \geqslant 0$ and strictly positive if A is indecomposable and if the diagonal elements of A are positive.

Proofs of the convergence have been given, among others, by Varga,[120] Bear, Jorgenson and Wagner,[121] and by Faddeev and Faddeeva.[122] The convergence of the series provides a sufficient condition for the nonsingularity of $(I - A)$.

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[120] Varga, *op. cit.,* pp. 82-83.

[121] Bear, Jorgenson, and Wagner, *op. cit.,* 64.

[122] D. K. Faddeev and V. N. Faddeeva, *Computational Methods of Linear Algebra,* translated by Robert C. Williams (San Francisco: W. H. Freeman and Co., 1963), p. 113.

Thirdly, the *Minkowsky-Leontief* condition simply refers to the properties of a Minkowsky-Leontief matrix itself, in which a given element a_{ij} is nonnegative and each column sum is less than unity. These simple conditions guarantee the nonsingularity of the corresponding matrix, $(I - A)$, as proved by Minkowski,[123] Woodbury,[124] and Wong and Morgenstern.[125]

Fourthly, the *dominant diagonality* condition refers to the Hadamard-McKenzie results, mentioned earlier, that in order for an open input-output system to have a unique nonnegative solution the Leontief matrix must have a *dominant* (or *quasi-dominant*) diagonal.

The most important point to underline here is that these four sets of necessary and sufficient conditions are equivalent. Mathematically, they are interlocked, in the sense that any one of them directly implies the others. Since many references have already been given to the proofs of these equivalent conditions, it is only necessary here to highlight a few points that are economically interesting. The first point concerns the positivity of the principal diagonals of the Leontief matrix and its economic implications. As we know, the Hawkins-Simon condition(s) derive from this mathematical property. Secondly, we will give a proof of the *convergence* condition and show how it leads to the *matrix multipliers* concept (i.e., the Leontief inverse, $(I - A)^{-1}$) which is of great economic importance.

1. Positivity of the Principal Minors of the Leontief Matrix and the Hawkins-Simon Condition(s)

Let A be an nxn ($n \geqslant 2$) Minkowski-Leontief (i.e., *technological* coefficients) matrix. Then, the associated characteristic equation can be written as

$$\det(\lambda I - A) = |\lambda I - A| = 0, \tag{27}$$

which is a polynomial in λ of degree n, having n roots $\lambda_1, \lambda_2, ..., \lambda_n$. We know that one of these roots, λ_m, is the maximal characteristic root, such that $\lambda_m > \lambda_i$, where λ_i is any other characteristic root, with the inequality becoming an equality when A is imprimitive.

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[123]Minkowski, *loc. cit.*

[124]Woodbury, *loc. cit.*

[125]Wong and Morgenstern, *op. cit.*, 225-237.

In short, we can see that det $(\lambda I - A)$ is equal to zero for all characteristic roots λ_i of A, including λ_m. For example, for λ_m we have:

$$(28) \qquad |\lambda I - A| = \begin{vmatrix} \lambda_m - a_{11} & - a_{12} & \cdots & - a_{1n} \\ - a_{21} & \lambda_m - a_{22} & \cdots & - a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ - a_{n1} & - a_{n2} & \cdots & \lambda_m - a_{nn} \end{vmatrix} = 0.$$

It can thus be seen that in order for the determinant of the characteristic equation to be positive, λ_m must be replaced by a number ω, such that $\omega > \lambda_m$. Since λ_m of A is equal to or greater than the greatest diagonal element in A, then replacing λ_m by ω in Eq. (28) will guarantee positive elements on the principal diagonal (i.e., $\omega - a_{11}, \omega - a_{22}, \dots, \omega - a_{nn}$ will be positive). In general, then, the determinant of an M-matrix, such as the Leontief matrix, is positive for $\omega > \lambda_m$. Consequently, for $\omega > \lambda_m$, the principal minors are all positive:

$$(29) \qquad \omega - a_{11} > 0; \quad \begin{vmatrix} \omega - a_{11} & - a_{12} \\ - a_{21} & \omega - a_{22} \end{vmatrix} > 0; \quad \begin{vmatrix} \omega - a_{11} & - a_{12} & - a_{13} \\ - a_{21} & \omega - a_{22} & - a_{23} \\ - a_{31} & - a_{32} & \omega - a_{33} \end{vmatrix} > 0; \text{ etc.},$$

for all a_{ii}.In a Leontief matrix, $(I - A)$, ω is equal to unity. Since λ_m of a Minkowski-Leontief matrix is less than unity (i.e., $\lambda_m = \rho(A) < 1$), then the condition $\omega > \lambda_m$ is met in the case of the Leontief matrix and all of its principal minors are positive.

In proving the positivity of the principal minors of $(I - A)$, Hawkins and Simon[126] assumed, implicitly, that $(I - A)$ is decomposable. Reducing it to an upper triangular form, they essentially argued that the positivity of the diagonal elements of the triangular matrix guarantees the positivity of its principal minors, and that since these minors are equal to the corresponding minors of the original matrix $(I - A)$, then the minors of $(I - A)$ are positive.

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[126] Hawkins and Simon, *loc. cit.*

Their decomposability assumption, of course, unnecessarily limits the generality of their proof.

The Hawkins-Simon condition(s), as given by Dorfman, Samuelson and Solow,[127] consist of two conditions: (i) the diagonal elements of the Leontief matrix must be positive (i.e., $1 - a_{11} > 0$, $1 - a_{22} > 0$, ..., $1 - a_{nn} > 0$), and (ii) the principal minors of the Leontief matrix must be positive. Actually, the first condition is a necessary part of the second, as clearly demonstrated in the Hawkins-Simon proof itself. Therefore, we shall henceforth refer to the Hawkins-Simon condition in the singular, referring only to the second condition that the principal minors of the Leontief matrix must be positive.

The condition that all principal minors of a Leontief matrix must be positive means, in economic terms, that the group of industries covered in each minor must be *self-sustaining,* that is, they must be capable of supplying more than their own direct and indirect needs in order to produce their own products.[128] Put somewhat differently, no group of industries should be *self-exhausting*, that is, the direct and indirect cost of no good in terms of itself should exceed unity. This means, in terms of the familiar coal example, that if a ton of coal requires or contains, directly *and* indirectly, more than one ton of coal, self-contained production is not viable.[129]

2. Convergence of the Power (*Multiplier*) Series to the Leontief Inverse and the *Matrix Multipliers* Concept.

The Minkowski-Leontief (i.e., *technological* coefficients) matrix, as we know, is both nonnegative (or positive in a more restricted sense) and meets the Cauchy-Hadamard condition.

$$\lim_{p\to\infty} \left\| A^p \right\|^{1/p} < 1, \tag{30}$$

where $\|A\|$ means the maximum value of the individual column sums in the A matrix.

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[127] Robert Dorfman, Paul A. Samuelson, and Robert M. Solow, *Linear Programming and Economic Analysis* (New York: McGraw-Hill Book Co., Inc., 1958), p. 215.

[128] Hawkins and Simon, *op. cit.,* 248.

[129] Dorfman, Samuelson, and Solow, *loc. cit.*

If this condition is valid, then the Cauchy-Hadamard theorem is also valid, according to which

$$(31) \qquad (I - A)^{-1} = I + A^*$$

where $\quad A^* = A + A^2 + A^3 + \dots A^p + \dots$

For the series

$$(32) \qquad I + A + A^2 + A^3 + \dots + A^p + \dots$$

to converge, it is necessary and sufficient that $A^p \to 0$ for $p \to \infty$. In this case,the *sum* of the series (32) is equal to $(I - A)^{-1}$. In order for $A^p \to 0$ for $p \to \infty$, it is necessary and sufficient that all characteristic roots of the matrix A be less than one in modulus.[130] It is clear that if $\rho(A) < 1$, $(I - A)$ is nonsingular, since then the associated characteristic root of $(I - A)$ is equal to $1 - \rho(A)$, such that $1 - \lambda_m(A) = \lambda_m(I - A) \neq 0$.

We can now give a proof of the convergence.[131] Let $\rho(A) < 1$, and $A^p \to 0$. The necessity of this condition is obvious. It will be shown that this is also sufficient. Since all characteristic roots of A are less than one in modulus, then, as we have just pointed out, $| I - A | \neq 0$, and therefore $(I - A)^{-1}$ exists.

We now consider the identity

$$(33) \qquad (I + A + \dots + A^p)(I - A) = I - A^{p+1}.$$

Post-multiplying both sides by $(I - A)^{-1}$ we have

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[130] For a proof of this condition, see, for example, Faddeev and Faddeeva, *op. cit.,* p. 111-112.

[131] *Ibid.* The proof given by Faddeev and Faddeeva has been modified here in order to improve its clarity.

(34) $$(I + A + \ldots + A^p)\ (I - A)\ (I - A)^{-1} = (I - A^{p+1})\ (I - A)^{-1}$$

which yields

(35) $$I + A + \ldots + A^p = (I - A)^{-1} - A^{p+1}\ (I - A)^{-1}.$$

This can be seen easily, since

(36) $$(I - A)\ (I - A)^{-1} = (I - A)^{-1}\ (I - A) = I.$$

Further, if we let $M = (I - A)^{-1}$ and $H = A^{p+1}$, then we have

(37) $$(I - A^{p+1})\ (I - A)^{-1} = (I - H)\, M = IM - HM = (I - A)^{-1} - A^{p+1}(I - A)^{-1}.$$

Hence, it follows that for $p \to \infty$,

(38) $$I + A + \ldots + A^p = (I - A)^{-1},$$

since $A^{p+1} \to 0$ for $p \to \infty$, and $A^{p+1}\ (I - A)^{-1}$ becomes zero.

Finally, we can determine the deviation of the Leontief inverse obtained through multiplier (power) series expansion from the inverse of full accuracy as follows: Since $\| A \| < 1$, then

(39) $$\| (I - A)^{-1} - (I + A + \ldots + A^p \| \leqslant \frac{\| A \|^{p+1}}{1 - \| A \|}\ .$$

This can be proved as follows. We have

(40) $$(I - A)^{-1} - (I + A + \ldots + A^p) = A^{p+1} + A^{p+2} + \ldots$$

This implies that

$$(41) \qquad \| (I - A)^{-1} - (I + A + \ldots + A^{p}) \| \leqslant \| A \|^{p+1} + \| A \|^{p+2} + \ldots$$

$$\leqslant \frac{\| A \|^{p+1}}{1 - \| A \|}$$

If we determine in advance that the error must be smaller than 1/100, this means that the following conditions must be valid for the number of *rounds* that should be followed in the power series expansion process. That is, if

$$(42) \qquad \frac{\| A \|^{p+1}}{1 - \|A\|} < \frac{1}{100}$$

then we have

$$(43) \qquad \| A \|^{p+1} < \frac{1}{100} (1 - \| A \|)$$

or

$$(44) \qquad (p + 1) \log \| A \| < \log \frac{1}{100} + \log (1 - \| A \|)$$

which finally yields[132]

$$(45) \qquad p \cong \frac{\log \frac{1}{100} + \log (1 - \| A \|)}{\log \| A \|} - 1$$

where p denotes the number of *rounds* that must be considered in order to guarantee a deviation (error) of less than 1/100 between an inverse of full accuracy and an approximated inverse using power series expansion. The number of *rounds* that should be followed is affected by the absolute value of the maximum column sum. Generally, the smaller the

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[132]This formulation is slightly different from those given by earlier writers. See V. Nyitrai, "Inversion of the Input-Output Table," in O. Lukács, *et al.* (eds.) *Input-Output Tables, Their Compilation and Use* (Budapest: Akadémiai Kiadó, Publishing House of the Hungarian Academy of Sciences, 1962), pp. 91-100. Also see Frederick V. Waugh, "Inversion of the Leontief Matrix by Power Series," Econometrica, XVIII (1950), 142-154; and Berger and Saibel, *op. cit.,*158 and 164.

maximum column sum, the smaller will be the number of *rounds* that should be followed to achieve the same level of accuracy.

The economic interpretation of the convergence of the power series $I + A + \ldots + A^p + \ldots$ to the Leontief inverse can be given as follows. If final demand is as specified by the column vector Y, then the total production required from every sector in order to satisfy this exogenous demand consists, first, of the final demand vector itself (IY), second, of the inputs needed to produce this final demand (AY), third, of the inputs needed to produce these inputs ($A^2 Y$), and so on.[133] Thus, every *round* is of a distinct value to the economist. The power series expansion process makes possible to examine the intermediate phases through which any given element in the Leontief inverse has finally obtained that value. That is to say, the chain reaction or *multiplier* process set into motion by a given rise in exogenous demand for final consumption can be traced through all of its various stages as it works itself out in the system until the system theoretically achieves a general equilibrium after every industry's demand generated by the initial rise in final demand has been satisfied. The inverse of the Leontief matrix represents this final equilibrium.

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[133]C. B. Tilanus, *Input-Output Experiments, The Netherlands 1948-1961* (Rotterdam University Press, 1966), p. 11.

APPENDIX B

RELATED TOPICS IN INPUT-OUTPUT ANALYSIS

A. INTRODUCTION

This appendix contains a discussion of a set of topics in input-output analysis the inclusion of which in Chapter I would have made it unduly long and would have hindered the continuity of the text. These topics consist of (a) the inverse of the Leontief matrix and *matrix multipliers*, (b) the relationship between final demand, sectoral output requirements, and value added (income generation), (c) the substitution theorem, (d) the Leontief Paradox, (e) prices in the open input-output system, (f) the closed input-output model, (g) the dynamic input-output model, and (h) the relationship of the basic open model to linear programming.

In the literature on input-output analysis, these topics have never really been pulled together in the way they are presented here. Although these topics could have been completely omitted without any loss to the major area of concern of this dissertation, they are briefly discussed here as part of a general effort made in this dissertation to make it as self-contained as possible. The various symbols used here are the same as those explained in Chapter I.

B. THE INVERSE OF THE LEONTIEF MATRIX AND MATRIX MULTIPLIERS

Every element α_{ij} in the inverse Leontief matrix $(I-A)^{-1} = [\alpha_{ij}]$ states the total *direct* and *indirect* (i.e., total) output required from industry i by the entire economy that is directly attributable to a unit of exogenous demand (i.e., a dollar's worth of demand) for the product of industry j. As it is often misunderstood, this does *not* mean that α_{ij} represents the direct and indirect inputs required from industry i *by* industry j in order for industry j to produce one unit (i.e., one dollar's worth) of its own output that is exogenously demanded for final consumption. The important point is that α_{ij} simply stands for that amount of *extra* output required from industry i for which industry j is directly and indirectly responsible when industry j has to deliver an extra dollar's worth of its own product to the final demand sectors for final consumption. To be sure, part of α_{ij} is actually required for intermediate consumption by industry j itself, as industry j attempts to produce one extra dollar's worth of its own output that is exogenously demanded. But the remainder is consumed by all other industries so that they can deliver to each other the necessary extra outputs that are generated in a number of *rounds* as a consequence of initially stipulating an extra dollar's worth of exogenous demand for industry j's product.

Seen in its entirety, then, the matrix $(I-A)^{-1} = [\alpha_{ij}]$ represents the total direct and indirect output required from every producing sector per unit of final demand for each (consuming) sector's output. This can be expressed mathematically as follows:

$$(1) \qquad X = I\,(I-A)^{-1} = (I-A)^{-1}\ I$$

where I stands for the diagonal matrix Y, showing *one dollar's worth* of final demand for the product(s) of each (consuming) sector. The total direct and indirect output required from any producing (row) sector i (i.e., x_i) in order to fulfill *one dollar's* worth of final demand imposed simultaneously for the products of every single (column) sector is defined by

$$(2) \qquad x_i = \sum_{j=1}^{n} \alpha_{ij}.$$

By the same token, total direct and indirect output required from *all* producing (row) sectors in order to fulfill *one dollar's worth* of final demand for the products of a given column sector is is defined by

$$(3) \qquad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \alpha_{ij}$$

which can be called an *aggregate sectoral multiplier* for a given column sector j.

The inverse of the Leontief inverse matrix $(I-A)^{-1} = [\alpha_{ij}]$ is called the *matrix multiplier*, the *multi-sector multiplier*, or simply *matrix multipliers*. The term *matrix multiplier* was first introduced by Goodwin,[1] who applied it to a concept that is, in an economic sense, more closely related to the Keynesian multiplier which has only a formal analogy to *matrix multipliers* represented by the inverse of the Leontief matrix. The subject of static and dynamic matrix multipliers, and the relationship between matrix multipliers and the *simple* or Keynesian multiplier model have been investigated by many writers, for example by Chipman, [2] Solow,[3] Chakravarty,[4] Hagger,[5] Amato,[6] and Bear,[7] just to mention a few.

Post-multiplying the *matrix multipliers* $(I-A)^{-1} = [\alpha_{ij}]$ for a given base year by the final demand vector Y exogenously derived for another year in the future, expressed in base year prices, then results in

$$(4) \qquad \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{bmatrix} \alpha_{11}y_1 + \alpha_{12}y_2 + \ldots + \alpha_{1n}y_n \\ \alpha_{21}y_1 + \alpha_{22}y_2 + \ldots + \alpha_{2n}y_n \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{n1}y_1 + \alpha_{n2}y_2 + \ldots + \alpha_{nn}y_n \end{bmatrix}$$

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[1] R.M. Goodwin, "The Multiplier as Matrix", *The Economic Journal.* LIX (December, 1949), 537-555.

[2] John S. Chipman, "The Multi-Sector Multiplier," *Econometrica,* XVIII (1950), 355-374; and "Professor Goodwin's Matrix Multiplier," *The Economic Journal,* LX (December, 1950), 753-770.

[3] Robert M. Solow, "A Note on Dynamic Multipliers," *Econometrica,* XIX, 3 (July, 1951), 306-316.

[4] Sukhamoy Chakravarty, "The Matrix Extension of the Multiplier," *The Indian Economic Review,* III, 3 (February, 1957), 41-52.

[5] A. Hagger, "Matrix-Multiplier Analysis: An Exposition," *Yorkshire Bulletin of Economic and Social Research,* X, 1 (June, 1958), 19-34.

[6] Vittorio Amato, "Rappresentazione matriciale del 'multiplocatore' dinamico," *L'industria Rivista di Economia Politica,* 3 (1961), 367-379.

[7] D.V.T. Bear, "The Matrix Multiplier and Distributed Lags," *Econometrica,* XXXI, 3 (July, 1963), 514-529.

where the column vector $X = (x_i)$ is the solution set, representing conditional point predictions of *equilibrium* sectoral output levels that will be required, directly and indirectly, to satisfy the exogenously specified *bill of goods*. If this solution set is to have any economic meaning, all elements of the vector $X = (x_i)$ must be nonnegative. If the economic system is to have a positive equilibrium corresponding to any nonnegative bill of goods (i.e., a solution set in which no sector produces a negative output), it is clear that every element in the inverse Leontief matrix $(I-A)^{-1}$ must be nonnegative. Conditions that guarantee the nonnegativity of the inverse Leontief matrix are also discussed at length in Appendix A.

What will be the rates of change of the solution values x_i with respect to the exogenous final demands $y_1, y_2, \ldots, y_n$? The general answer is given by

$$(5) \qquad \frac{\partial x_i}{\partial y_k} = \alpha_{ik} \qquad (i,k = 1,2,\ldots,n)$$

where, for k=i, the answer is simply α_{ii}. To see this, we can refer back to (4), the partial differentiation of which will produce a total of $n \mathrm{x} n = n^2$ comparative-static derivates:

$$(6) \qquad \begin{array}{llll} \dfrac{\partial x_1}{\partial y_1} = \alpha_{11}, & \dfrac{\partial x_1}{\partial y_2} = \alpha_{12}, & \ldots, & \dfrac{\partial x_1}{\partial y_n} = \alpha_{1n} \\ \dfrac{\partial x_2}{\partial y_1} = \alpha_{21}, & \dfrac{\partial x_2}{\partial y_2} = \alpha_{22}, & \ldots, & \dfrac{\partial x_2}{\partial y_n} = \alpha_{2n} \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ \dfrac{\partial x_n}{\partial y_1} = \alpha_{n1}, & \dfrac{\partial x_n}{\partial y_2} = \alpha_{n2}, & \ldots, & \dfrac{\partial x_n}{\partial y_n} = \alpha_{nn}. \end{array}$$

This is simply the expanded version of (5).

Reading (6) as n distinct columns, we may combine the n partial derivates in each column into a matrix (vector) derivative:

(7)
$$\frac{\partial x}{\partial y_1} \equiv \frac{\partial}{\partial y_1}\begin{pmatrix} x_i \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \vdots \\ \alpha_{n1} \end{pmatrix}, \quad \frac{\partial x}{\partial y_2} = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \\ \vdots \\ \alpha_{n2} \end{pmatrix}, \text{ etc.}$$

Since the column vectors in (6) are merely the columns of the matrix $[\alpha_{ij}]$, by further consolidation we can summarize the n derivates in a single matrix derivative $\partial x/\partial y$. Given $(x_i) = [\alpha_{ij}]\ (y_i)$, we can simply write

(8)
$$\frac{\partial x}{\partial y} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix} = (I-A)^{-1}$$

This is a compact way of denoting all the comparative-static derivates of the open input-output model.

A comparison between the two matrices A and $(I-A)^{-1}$ is of interest. An element in $(I-A)^{-1}$ should be at least as large as the corresponding element in A, the difference indicating the amount of *indirect* output required from the producing (row) sector per unit of final demand for the products of the column sector. Further, while A would be expected to be relatively *sparse* (i.e., with many zeroes, particularly when A is a large matrix), $(I-A)^{-1}$ would be expected to be *dense* (i.e., with relatively few zeroes). This would indicate, as one should expect from an economic standpoint, that with a few exceptions, all commodity-groups are consumed, directly or indirectly, in the production of any of the groups. Most of the elements in $(I-A)^{-1}$ should be between zero and unity, indicating that when one unit of a certain product-group is exogenously demanded for final consumption, the total production required of each commodity-group is less than one unit. The total production of a given commodity-group itself, however, must be necessarily at least one (unity), since in each case it must itself produce the one unit of output exogenously demanded from it for final consumption. As a rule, the diagonal elements in $(I-A)^{-1}$ exceed unity, reflecting the fact that the production for final consumption of one unit of a commodity-group generally requires further production by the same sector, in order to satisfy the direct and indirect input requirements in the system already set into motion.

C. THE RELATIONSHIP BETWEEN FINAL DEMAND, SECTORAL OUTPUT REQUIREMENTS, AND VALUE ADDED (INCOME GENERATION)

As noted above in (1.6), a given primary factor input w_{kj} (e.g., labor per unit of an industry's output x_j is represented by the coefficients b_{kj} ($k=1,2,\ldots,m$; $j=1,2,\ldots,n$), such that $w_{kj} = b_{kj}\, x_j$. Total factor inputs of an industry can then be simply expressed as

$$\sum_{k=1}^{m} w_{kj} = \sum_{k=1}^{m} b_{kj}\, x_j,$$

or simply as $w_j = b_j x_j$, where w_j stands for *value added* or *income generated.* Letting $W = (w_j)$ be a 1xn row vector and $\hat{b} = (b_j)$ be an nxn diagonal matrix and substituting $(I-A)^{-1}\; Y$ for $X = (x_j) \equiv (x_i)$, we have

(9) $$W = \hat{b}\,(I-A)^{-1}\, Y$$

which expresses the relationship between the final demand vector Y and value added (income generation) in each industry. This equation can be used to determine the total sectoral primary factor input requirements (income generation), corresponding to a given *bill of goods* or final demand vector.

Of course, we can just as easily use the entire mxn matrix consisting of b_{kj}, $B = (b_{kj})$, in (9), in which case $B\,(I-A)^{-1}$ would be an mxn matrix. Then, a given element in this matrix would express the total (direct and indirect) demand for a particular factor input (e.g., labor) per unit of final consumption demand for the products of a given column sector.

D. THE SUBSTITUTION THEOREM

The use of *fixed* coefficients in the model is usually thought to rule out the possibility of substitution, as assumed in the classical Clark-Wicksteed-Walras theory of production and general equilibrium. To put it somewhat differently, the implication of Leontief's system is that even if several production processes were available to an industry, only one of them would actually be observed. Thus, the economy would always operate as if it knew only one set of input ratios for each commodity. This does not mean that changes in technological information will not lead to changes in input coefficients. It does mean, however, as Samuelson showed in his substitution theorem,[8] that in the Leontief model which has only one scarce factor (i.e., labor), only one activity would ever be used for producing each commodity, no matter how final demand changed. Even if substitution were *physically* possible, it would be ruled out on *economic* grounds, since a change in wages would leave the relative factor prices unchanged due to the fact that everything is *congealed* labor in the Leontief system.

E. THE LEONTIEF PARADOX

In two articles[9] which led to a long controversy, Leontief computed and compared the amount of capital and labor required, directly and indirectly, to produce in the United States two composite goods, each worth one million dollars, and each representing, respectively, exports and competitive imports of the United States. Leontief found that an average million dollar's worth of U.S. exports embodied considerably less capital and somewhat more labor than would be required to replace from domestic production an equivalent amount of competitive imports.[10] That is, the participation of the United States in the

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[8] See Paul A. Samuelson, "Abstract of a theorem Concerning Substitutability in Open Leontief Models," in T. C. Koopmans (ed.), *Activity Analysis of Production and Allocation,* Proceedings of a Conference, Cowles Commission for Research in Economics (New York: John Wiley and Sons, Inc., 1951), pp. 142-146.

A full discussion of the substitution theorem can be found in Robert Dorfman, Paul A. Samuelson, and Robert M. Solow, *Linear Programming and Economic Analysis* (New York: McGraw-Hill Book Co., 1958), pp. 224-227 and pp. 248-252.

[9] Wassily Leontief, "Domestic Production and Foreign Trade: The American Capital Position Re-examined," *Economia-Internazionale* (February 7, 1954), 9-32; and "Factor Proportions and the Structure of American Trade: Further Theoretical and Empirical Analysis," *The Review of Economics and Statistics,* XXXVIII (November, 1956), 386-407.

[10] Leontief, *op. cit., 24 [Domestic Production...]*

international division of labor seemed to be based on its specialization of labor intensive, rather than capital intensive, lines of production. His conclusion, then, was:

> ... in other words, this country resorts to foreign trade in order to economize its capital and dispose of its surplus of labor, rather than vice versa. The widely held opinion that – as compared with the rest of the world – the United States' economy is characterized by a relative surplus of capital and a relative shortage of labor proves to be wrong. As a matter of fact, the opposite is true.[11]

In a recent article, Brecx[12] maintains that in order to assess the economy or dis-economy of capital and labor experienced by the United States through its participation in international trade, a slightly different procedure must be used. According to Brecx, one should compute and compare the quantities of capital and labor directly and indirectly required to produce in the United States one million dollar's worth of competitive imports *not* with exports of the same value *but* with the value of exports which equilibrates exactly the balance of payments.[13] On this basis, he proceeds to refute Leontief's conclusions.

F. PRICES IN THE OPEN INPUT-OUTPUT SYSTEM:

Let the price per unit of a homogeneous good produced by given sector be p_i, such that p_i is just equal to the total outlays incurred by each sector in the production of a unit of its homogeneous good. Then we have:[14]

$$
\begin{aligned}
p_1 &= a_{11} p_1 + a_{21} p_2 + \ldots + a_{n1} p_n + w_1 \\
p_2 &= a_{12} p_1 + a_{22} p_2 + \ldots + a_{n2} p_n + w_2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
p_n &= a_{1n} p_1 + a_{2n} p_2 + \ldots + a_{nn} p_n + w_n
\end{aligned} \tag{10}
$$

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[11] *Ibid.*

[12] Paul Brecx, "Leontief's Paradox," *The Review of Economics and Statistics,* XLIX, 4 (November, 1967), 603-607; followed by a short reply by Leontief.

[13] *Ibid.,* 603.

[14] See Wassily Leontief, *Input-Output Economics* (New York: Oxford University Press, 1966), pp. 143-145.

where w_j denotes *value added* by each sector and where the *technological* coefficients a_{ij} are measured in their natural (physical) dimensions.

Clearly, these price equations (10) can be rewritten in the following more familiar way:

$$
\begin{aligned}
(1-a_{11})\,p_1 - a_{21}\,p_2 - \dots - a_{n1}\,p_n &= w_1 \\
-a_{12}\,p_1 + (1-a_{22})\,p_2 - \dots - a_{n2}\,p_n &= w_2 \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
-a_{1n}\,p_1 - a_{2n}\,p_2 - \dots + (1-a_{nn})\,p_n &= w_n
\end{aligned} \tag{11}
$$

which, in matrix form, can be written as[15]

$$(I - A)'\ P = W \tag{12}$$

where $(I - A)'$ is the transpose of the Leontief matrix, and P and W are, respectively, the price (per unit) and value added (per unit) vectors. Hence, if we know the value added by each sector per unit of its output, we can obtain the *price solution* by solving the equation system (12):

$$P = [(I - P)']^{-1}\ W \tag{13}$$

G. THE CLOSED-STATIC INPUT-OUTPUT MODEL

Early in his work, Leontief formulated his closed-static input-output model which, in summary, amounts to treating the final demand sectors of the open model as endogenous sectors and expressing the system as a set of homogeneous linear equations. Such a summary statement, however, somewhat oversimplifies the complicated nature of the model as fully explained by Leontief in his earlier volume on the structure of the American economy.[16]

The closed-static model can be developed for exposition in two alternative ways: first, under the assumption of stationary equilibrium (i.e., an hypothetical state with no saving and investment), and second, under the assumption that saving and investment do exist.

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[15] George Hadley, *Linear Programming* (Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1962), pp. 490-492.

[16] Wassily Leontief, *The Structure of the American Economy, 1919-1939* (New York: Oxford University Press, Second Edition, 1951). See particularly the first two parts.

The various discussions of the closed model by other writers seldom go beyond the simplified treatment of the model as formulated under the assumption of stationary equilibrium.[17] A complete mathematical restatement of the model has been developed by this writer, but it is too long to present here. The remarks below will thus be confined only to the closed model formulated under the assumption of stationary equilibrium.[18]

It will be recalled that in the open model, households, government, investment, and exports make up the final demand sectors and are treated exogenously. By contrast, in the closed model, they are all made endogenous to the model. Hence, households now become an *industry,* furnishing its product (services) to other industries in terms of labor, in return for consumer goods (i.e., inputs into the household *industry*). Similarly, government is now treated as an industry which makes payments to other sectors of the economy for the goods and services it purchases and which provides its own services to other sectors the costs of which are met by other sectors by the various taxes that they have to pay. Finally, foreign trade is treated as an industry whose *purchases* consist of exports and whose product is imports.

Mathematically, the closed-static input-output system reduces to

$$(I - A)\,X = 0 \tag{14}$$

which is the same as (1.13) in Chapter I, except for the fact that now the final demand vector Y is zero (i.e., it is a *null* vector). This linear homogeneous system has the trivial solution $X = (x_i)$ in which every x_i is zero. Of course, the existence of a nontrivial solution is of far greater interest. A linear homogeneous system has a nontrivial solution if and only if the number of unknowns (columns) is greater than the rank of the coefficient matrix.[19]

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[17] See, for example, the following:

Dorfman, Samuelson, and Solow, *op. cit.,* pp. 245-248;

Robert E. Kuenne, *The Theory of General Economic Equilibrium* (Princeton, N. J.: Princeton University Press, 1963), pp. 384-386;

Harlan S. Smith, "Uses of Leontief's Open Input-Output Models," in Tjalling C. Koopmans (ed.), *Activity Analysis of Production and Allocation,* Proceedings of a Conference, Cowles *Commission for Research in Economics* (New York: John Wiley Sons, Inc., 1951), pp. 132-140.

[18] Kirkor Bozdogan, "A Mathematical Restatement of Leontief's Closed Input-Output Model" (unpublished paper, Massachusetts Institute of Technology, March, 1968), 13 pp.

[19] See, for example, Sam Perlis, *Theory of Matrices* (Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1958), p. 47.

Further, given the solution $X = (x_i)$, $X^* = (x_i^*)$ is also a solution where $x_i^* = \lambda_i x_i$. That is, any linear combination of any solution of $(I - A) X = 0$ is itself a solution. This means that the closed-static input-output model under the assumption of stationary equilibrium can be solved only for *relative* sectoral output levels measured in physical terms.[20]

H. THE DYNAMIC INPUT-OUTPUT MODEL

Leontief's dynamic general equilibrium system is described by the following set of n linear differential equations

$$x_i(t) - \sum_{j=1}^{n} a_{ij} x_j(t) - \sum_{j=1}^{n} c_{ij} \dot{x}_j(t) = y_i(t) \tag{15}$$

where

$x_i(t)$ represents the rate of output of good i produced by the ith sector of the economy at the point of time t;

$\dot{x}_j(t)$ is the first derivative (i.e., rate of change) of x_i at time t;

a_{ij} are the familiar *technological* coefficients indicating the the amount (value) of product i absorbed by sector j on *current account* per unit of sector j's own output;

c_{ij} is the capital coefficient which shows how large a stock of the output of sector i is required by sector j per unit of the flow of its respective output (per unit of time); and

$y_i(t)$ represents the exogenous demand for good i which in an open system is considered to be a given function of time.[21]

A detailed discussion of the dynamic system can be found in what Solow[22] has called *locus classicus,*[23] as well as in the Dorfman-Samuelson-Solow volume.[24] In a series of articles,

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[20] Kuenne, *op. cit.,* p. 374.

[21] If $y_i(t) = 0$ at all times, the dynamic system is called *closed.*

[22] Robert M. Solow, "Competitive Valuation in a Dynamic Input-Output System," *Econometrica,* XXVII, 1 (January, 1959), 30.

[23] Leontief, *et al., op. cit., pp.* 55-90 *[Studies in the Structure...].*

[24] Dorfman, Samuelson, and Solow, *op. cit.,* pp. 283-300.

Morishima[25] has made extensive contributions to the literature on dynamic input-output systems. A formulation of the dynamic system in terms of difference equations is given by Wurtele.[26] The debate between Sargan and Leontief on the stability of the open dynamic system makes interesting reading.[27] Finally, the application of the open dynamic system to the problem of making long-range projections of economic growth is given by Leontief[28] and its application to economic planning is explored by Mathur.[29]

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[25] See, for example, the following:
Michio Morishima, "Prices, Interest, and Profits in a Dynamic Leontief System," *Econometrica,* XXIV, 3 (July, 1958), 358-380; and "Some Properties of a Dynamic Leontief System with a Spectrum of Techniques," *Econometrica,* XXXVII, 4 (October, 1959), 626-637.

[26] Zivia S. Wurtele, "A Note on Some Stability Properties of Leontief's Dynamic Models," *Econometrica,* XXVII, 4 (October, 1959), 672-675.

[27] J. D. Sargan, "Lags and the Stability of Dynamic Systems: A Reply," *ibid.,* 670-673;
Wassily Leontief, "Lags and the Stability of Dynamic Systems: A Rejoinder," *ibid.,* 674-675.
J. D. Sargan, "The Instability of the Leontief Dynamic Model," *Econometrica,* XXVI, 3 (1958), 381-392;
Wassily Leontief, "Lags and the Stability of Dynamic Systems," *Econometrica,* XXIX, 4 (October, 1961), 659-669;

[28] Wassily Leontief, "An Open Dynamic System for Long-Range Projection of Economic Growth," *Artha Vijñana,* IX, 3-4 (September-December, 1967), 370-390.

[29] P. N. Mathur, "An Application of Dynamic Input-Output Model," *ibid.,* 391-411.

I. THE RELATIONSHIP OF THE BASIC OPEN MODEL TO LINEAR PROGRAMMING

The standard formulation of the basic open model in (1.13) in Chapter I

$$(I - A)\,X = Y \tag{16}$$

contains no hint of the linear programming type of optimization. There is no objective function to optimize. Further, no inequalities appear in the basic formulation, although the set of equations does stipulate *constraints* on the output level of each sector (i.e., each sector should produce enough output to satisfy the total demand, both for intermediate and final consumption).

The same input-output model can be formulated as a linear programming problem. First of all, the output levels required of each sector should be *no less than* (rather than equal to) the total demand for it. Consequently, we can change the equality in (7) into an inequality

$$(I - A)\,X \geqslant Y. \tag{17}$$

In order to insure against the $>$ part of the $\geqslant$ sign to get out of hand, a minimization requirement should accompany this inequality. Assuming labor to be the only primary input, for example, we can seek to minimize total labor requirements while producing at least the bill of goods Y. Letting b_j stand for the amount (value) of labor required per unit of sector j's output, we would like to minimize

$$L = \sum_{j=1}^{n} b_j\, x_j = (b_1\ b_2\ \ldots\ b_n) \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = B'X \tag{18}$$

where $B' = (b_j)$ is a row vector. Finally, we know that output levels can never be negative, and will want to impose the restriction $x_i > 0$. In this way, the open system can be reformulated in the following mathematically equivalent form:

(19) $$\begin{aligned}&\text{Minimize } L = B'X\\ &\text{Subject to } (I - A)X \geqslant Y\\ &\text{and } X \geqslant 0\end{aligned}$$

which is the standard linear programming problem,[30] for which the unique optimal solution is given by

(20) $$L = B'(I - A)^{-1}Y$$

where the optimal output vector is necessarily that given by the regular open system $X = (I - A)^{-1}Y$. This can be explained by the fact that if labor is an indispensable input for every good, then the output vector with the least labor requirement must by necessity be that which contains no excess demand over total demand.

(21) The dual of (19) is

$$\begin{aligned}&\text{maximize } Z = Y'\omega\\ &\text{Subject to } (I - A)'\ \omega \leqslant B\\ &\qquad\qquad\qquad\quad\ \omega \geqslant 0.\end{aligned}$$

The unique optimal solution to the problem is given by

(22) $$\omega = [(I - A)']^{-1}B$$

where $\omega = (\omega_i)$ is a vector of accounting prices (i.e., shadow prices) which reflect the *true* prices of final consumption goods in terms of labor costs and which would exist if the economy operated under pure competition and in a state of long-run equilibrium, with labor as the only primary input.

The simple linear programming formulation given here can be extended by introducing additional primary factors and by stipulating constraints on them. Further, capacity constraints can be imposed on each industrial sector. These extensions can be found in Chenery and Clark,[31] Kurihara,[32] and Stone.[33]

It is possible to note at least four differences between the open Leontief system and linear programming. *First,* linear programming enables us to choose one solution as better than another. This choice is not present in the open Leontief system. *Secondly,* under linear programming there exist alternative ways of producing the same output. Thus, the one good-one industry input-output restriction is no longer necessary. In this way, we no longer have to restrict ourselves to square *technology* matrices; they can be made rectangular. Consequently, the joint products problem disappears. *Thirdly,* in the linear programming formulation, the primary factor inputs are made part of the model, since any solution must satisfy resource limitations as well as requirements for final consumption. *Lastly,* the constraints in the linear programming formulation consist of inequalities, rather than of equalities, thus allowing for the non-use of some resources.[34]

Finally, it must be noted that the *technological* coefficients matrix of the open Leontief system plays a major role in the linear programming formulation. This is why the experiments reported in Chapter IV have important implications not only for the use of input-output models in making predictions but also for their use in programming for economic development.

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[30] For a similar formulation, refer to Hadley, *op. cit.,* p. 491. Also see Alpha C. Chiang, *Fundamental Methods of Mathematical Economics* (New York: McGraw-Hill Book Co., Inc., 1967), pp. 639-644. A more extensive discussion can be found in Hollis B. Chenery and Paul G. Clark, *Interindustry Economics* (New York: John Wiley and Sons, Inc., Fourth Printing, 1965), pp. 81-136.

[31] Chenery and Clark, *op. cit.,* p. 86.

[32] Kenneth K. Kurihara, *Macroeconomics and Programming* (London: George Allen and Unwin, Ltd., 1964), pp. 53-74.

[33] Richard Stone, *Input-Output and National Accounts* (Paris: The Organization for European Economic Co-operation, 1961), pp. 139-155.

[34] Chenery and Clark, *loc. cit.*

APPENDIX C

THE DEFINITION AND MEASUREMENT OF FINAL DEMAND SECTORS AND PRIMARY INPUTS

A. INTRODUCTION

This appendix is an extension of the discussion given in Chapter II on the conceptual and empirical problems faced in input-output model construction and the methods used to overcome them. Two topics are covered here: (1) the definition and measurement of final demand sectors, and (2) the definition and measurement of primary inputs (value added). As before, the discussion is based on the 1947 and 1958 input-output studies of the United States, and again, as before, the purpose is to develop a systematic and critical understanding of the methods used in input-output model construction to overcome a series of measurement problems.

B. THE DEFINITION AND MEASUREMENT OF FINAL DEMAND SECTORS

The input-output framework requires not only the recording of intersectoral flows (production and consumption of goods and services for intermediate use), but also final output (for consumption by final demand sectors), and payments to factors of production (value added, or primary inputs). By contrast, in national income and product accounts, only final output is recorded. All intermediate flows are netted out in order to avoid duplication in the measurement of production. In an input-output system, intermediate flows play the most important role, by yielding the *technological* coefficients matrix.

If the input-output system is fully integrated with national income and product accounts, as was the case in the 1958 Input-Output Study of the United States, the sum of the final demand sectors should equal gross national product (GNP) on the product side, while the *value added* row sum should equal GNP on the income side. When final demand sectors are fully integrated with national income and product accounts, each final demand sector corresponding to a particular GNP component must be specified as an nxl vector.[1]

The final demand sectors consist of (1) the household or personal consumption expenditures sector, (2) government purchases of goods and services (including, in the case of the United States, local, state, and federal government expenditures), (3) gross fixed private capital formation, (4) net inventory change, and (5) foreign trade (consisting only of exports in most tables). These five basic final demand sectors can be expanded into many more columns, by simply specifying the various components of each basic sector. For example, the household sector can be broken down into different columns each showing the consumption pattern of households in the various income categories. Likewise, the government sector can be divided into the various levels of government, with the federal government expenditures specified by type, such as defense and nondefense. Exports can be extended into a full matrix, designating in each column the country of destination. In other words, many opportunities do exist for showing final consumption demand in much greater detail. In analytical applications, however, they must be aggregated into a single sector (i.e., an nxl vector) for mathematical convenience.

The definition and measurement of final demand sectors present many difficult conceptual and empirical problems a full discussion of which would be beyond the scope of this appendix. The focus here, therefore, will be on the most important problems and the methods used to overcome them.

1. Household (Personal Consumption Expenditures) Sector[2]

Personal consumption expenditures (PCE), as defined in the 1958 Input-Output Study, consist of the value of goods and services purchased by individuals and nonprofit institutions

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[1] In the case of the *exports* column, the convention used is to obtain the difference between the sum of the *exports* column and the *imports* row(s) in order to estimate the *Net Foreign Trade* component of GNP.

[2] This part is largely based on the article by Simon. See Nancy W. Simon, "Personal Consumption Expenditures in the 1958 Input-Output Study", *Survey of Current Business,* XLIV, 10 (October, 1965), 7-11 and the Appendix (pp. 27-28).

rendering services to individuals, *plus* the value of certain imputed goods and services received by individuals as income by kind.[3] In the same study, the *commodity flow* method was used in estimating PCE. Also used in the estimation of national income and product account, this method enables the identification of all goods destined for personal consumption from the output records of farms, factories, etc., such that the flow of output is followed through the distribution channels and the costs of distribution are added to flows valued at producer's prices in order to arrive at the prices paid by the consumers. On a detailed basis, the GNP and PCE show important differences. For example, while the GNP consumer expenditures entries are classified in terms of functional categories (e.g., food expenditures, which is further classified as food purchased for off-premise consumption, purchased meals and beverages, food furnished to government-including military-employees and commercial employees, and food produced and consumed on farms), the PCE entries in input-output tables are classified by producing sectors (e.g., livestock and livestock products, other agricultural products, forestry and fishery products, food and kindred products, etc.). Further, the GNP personal consumption expenditures data are expressed in purchaser's prices, while the input-output PCE entries must be expressed in producer's prices (usually including, by convention, excise taxes levied on the producer). Distributive costs, such as transportation costs, trade margins (by convention, including retail excise taxes and sales taxes) are shown separately in the PCE column as purchases from these *margin* sectors.[4] The entry in the intersection of the trade row and the PCE column

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[3] *Ibid.*, p. 7. A list of the imputed items is given in footnote 4 as follows: (1) the space-rental value of owner-occupied houses (but not the purchase of new dwellings, which are considered capital goods as in the 1947 study; (2) the value of food, clothing, and housing furnished in kind to government (including military) and business employees; (3) food and fuel produced and consumed on farms; and (4) services rendered to individuals and nonprofit institutions by financial intermediaries (except insurance companies) without explicit charge.

[4] It must be remembered that some of these *margin* sectors directly sell services to the household sector (i.e., nonmargin purchases by the household sector), including the services that consumers buy directly from airlines, railroads, bus companies, etc., and consumer purchases of health insurance, bank services (in the form of bank service charges), etc. These nonmargin purchases from the *margin* sectors are entered in the PCE column in producer's prices. To these must be added the *margin* purchases in order to find total consumer expenditures absorbed by each of the *margin* sectors.

represents, for example, the aggregate cost of distributing all entries in the PCE column through trade channels.[5]

Each entry in the PCE column, excepting the *margin* entries, represents purchases by the consumers from any given domestic producing sector, where the *producing sector* refers to the primary products of a domestic industry, whether in fact produced by the primary industry or produced as a secondary product by other industries. It will be recalled that in the 1958 Study PCE entries also refer only to domestically produced commodities or services (excepting at least one inconsistency already indicated)[6], and that all household purchases of noncompetitive imports are shown in the *imports* row (Row 80A).

Some of the measurement problems can be identified by giving a few examples from the 1947 Study and by indicating a few of the differences that exist between the 1947 and the 1958 studies pertaining to both coverage and treatment.

In the 1947 Study, the precedent was established to include in the PCE column food produced and consumed on farms but to exclude the costs of farm operations. Purchases of houses by individuals or households for self-occupancy were treated as business investment and included in the gross fixed private capital formation column as investment. Rental payments covered both rents paid by tenants and imputed rents of households. Tenant paid rents, covering both contract rent and utilities, was more inclusive than the space rental concept used in the compilation of national income and product accounts. Most of the maintenance costs associated with residential buildings was charged as a cost to the rental industry, while the small outlay shown in the PCE column represented maintenance outlays by tenants not constituting a cost to the rental industry. Certain expenditures by individuals for travel in connection with their business activities were for the most part also included

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[5] The only nonmargin purchase from the trade sector is *tips*. See Simon, *op. cit.*, p. 12.

[6] Imported automobiles are shown as being purchased from the motor vehicles and equipment sector. This point has been raised earlier in discussing the treatment of imports.

in the PCE column. Similarly, business expenditures, such as on hand tools by carpenters, were included, as were some other expenditures by individuals in connection with their business activities.[7]

There were many differences between the 1947 and 1958 studies in both the coverage and treatment of consumption expenditures. For example, in the 1947 Study, poultry and meat slaughtered on farms, whether for sale or for home consumption, were shown as a purchase by households from one of the farming sectors. In the 1958 Study, these items were regarded as secondary products of farming and classified for distribution as part of the primary output of the food and kindred products sector. Milk processing was made subject to the same type of shift, by including it in the 1958 Study as part of the food and kindred products sector. Further, in 1947 eating and drinking places were treated as a separate industrial sector, while in 1958 they were treated as a trade margin. In addition, travel and entertainment expenditures were not divided between business and consumers in the 1947 Study, and all such purchases were considered to have been made by the household sector. In the 1958 Study, on the other hand, travel and entertainment expenditures were separated into business and consumer shares, and only the consumer part was shown as an entry in the PCE column. Lastly, nonlife insurance was measured in the 1947 Study as gross premiums earned[8], while in the 1958 Study it was measured as premiums earned less benefits paid.[9] Further, in the 1958 Study, remittances to foreigners in cash and in kind were taken out of PCE and was made a personal transfer to foreigners. Similarly, the payment by households of interest on personal debt was eliminated from PCE and was instead treated as a government purchase. Lastly, expenditures by Mexicans and West Indians working temporarily in the United States were excluded from PCE and treated as exports.

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[7] For some of these points, see Philip M. Ritz and Gabriel G. Rudney, *The 1947 Interindustry Relations Study, Industry Reports, General Explanations,* U.S. Department of Labor, Bureau of Labor Statistics, Division of Interindustry Economics, BLS Report No. 9 (March, 1953), p. 36.

[8] *Ibid.* Also see W. Duane Evans and Marvin Hoffenberg, "The Interindustry Relations Study for 1947", *The Review of Economics and Statistics,* XXXIV, 2 (May, 1952), III.

[9] Simon, *op. cit.,* 28.

There were many other differences between the two studies on the coverage and treatment of personal consumption expenditures. The few examples given here are not only illustrative of some of the differences in measurement that exist between the two studies but also provide a glimpse of the myriad of conceptual, empirical measurement, and classification problems that are universally faced in input-output analysis.

2. Government Sector

Total government expenditures on current account, represented by the sum of the government column(s), measure the gross input of the government sector in the economy. Government output is measured as the sum of the government row(s) (shown within the *primary inputs* or *value added* part of the input-output table) and consists of total receipts on current account, including both tax revenue and miscellaneous receipts.

In both the 1947 and 1958 input-output studies of the United States, government expenditures on current account were defined to include not only expenditures for goods and services used in ordinary government operations by both government agencies and corporations, but also expenditures for items that would, if they were in the private sector, be considered as capital expenditures.[10] Thus, no distinction is made in the government column entries between expenditures on current or capital account, although usually government purchases representing fixed investment are separated and clearly shown. Excluded from government expenditures are such items as acquisitions of land, current outlays of government enterprises, and strictly financial transactions (e.g., loans, payments of claims, etc.). Also excluded are government transactions of physical goods produced in previous time periods.[11]

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[10] Sidney A. Jaffe, "Final Demand Sectors", in Philip M. Ritz (ed.), *Input-Output Analysis, Technical Supplement,* Conference on Research in Income and Wealth (New York: National Bureau of Economic Research, Inc., 1954), Part 1, p. 28.

[11] See U.S., Department of Commerce, Office of Business Economics, National Income Division, *The 1958 Interindustry Relations Study,* Unpublished Preliminary Report (November, 1964), Appendix 2, p. 3; Irving H. Licht, "Government", in Philip M. Ritz (ed.), *Input-Output Analysis, Technical Supplement,* Conference on Research in Income and Wealth (New York: National Bureau of Economic Research, Inc., 1954), Part 2, p. 4; and Evans and Hoffenberg, *op. cit.,* 110.

Government enterprises, defined in the 1958 Study as public functions that cover over half of their current operating costs by the sale of goods and services to the general public,[12] were treated in both the 1947 and the 1958 input-output studies as intermediate sectors, and thus excluded from government accounts. In the 1947 Study, these included liquor monopolies, electric power plants and gas supply systems, industrial plants, all local public transit systems and administration of municipal airports,[13] commercial and financial enterprises (e.g., municipal radio stations, the U. S. Government Printing Office, the Federal Reserve Bank System, etc.), and schools, hospitals, and other public institutions. On the other hand, operations of the U. S. Post Office, local water works and sewage systems, certain important industrial functions of the U. S. Department of Defense (e.g., the operation of arsenals, ordnance plants, and naval shipyards), and financial activities of government corporations (e.g., Commodity Credit Corporation, Federal Deposit Insurance Corporation, Federal Land Banks, Federal Surplus Commodity Corporation, etc.) were included in the government sector.[14] The state and local government sectors, also as defined in the 1947 Study, included states, cities, counties, townships, and special districts except school districts which were made a part of the education *industry*.[15]

In the government column(s), all public new and maintenance construction (including force account) expenditures were treated in the 1947 Study as purchases from the respective construction sectors, rather than as purchases of a variety of different items (from different sectors) entering construction costs (e.g., the cost of materials, services, wages and salaries). Similarly, government expenditures on health and education were treated as purchases from the hospital and education sectors. Expenditures on equipment pertaining to government activities, such as that used in public construction, and in the operation of

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[12] Norman Frumkin, "Construction Activity in the 1958 Input-Output Study", *Survey of Current Business,* XLV, 5 (May, 1965), 13, footnote 2.

[13] Excluding canals and port facilities operated under public authority. See Licht, *op. cit.,* Part 2, p. 15.

[14] *Ibid.,* Part 2, pp. 15-19.

[15] Ritz and Rudney, *op. cit.,* p. 32.

public hospitals and schools, were charged to the government sector. Government interest payments on the public debt were defined in accrual rather than cash terms in order to maintain consistency with the national income and product accounts definitions, and were recorded on a net basis (i.e., *net interest paid by government*). Similarly, government unilaterals, as indicated earlier, were recorded on a net basis. Government payments of interest to social insurance funds and contributions to such funds (which can be considered as wage supplements in the same sense as employer contributions to social insurance) were included in intragovernment transactions, by regarding them as real costs to government for services rendered. The intragovernment entry also included payments of one government sector to another, such as federal grants-in-aid to the various states.[16]

3. Gross Private Capital Formation

Gross private capital formation or investment represents outlays for goods and services during the accounting period charged by business to capital account. In general, such outlays include capitalized new durable equipment, new private construction (including dwellings acquired for owner occupancy) and miscellaneous charges to capital formation. Net change in inventories is handled more conveniently as a separate sector. Capital formation within the government sector is also indicated separately. The gross capital formation or investment column in an input-output table shows the output of all capital goods for private use by every sector, regardless of the industry of destination actually making the investment. The counterpart of the investment column in an input-output table is the capital consumption allowances (depreciation) row, included within the *primary inputs* part of the table. The algebraic sum of the investment column and depreciation row totals yields net private capital formation.

The empirical measurement of investment for input-output purposes presents many difficulties. First, a decision must be made on what criterion to follow in classifying purchases on capital account as investment. As indicated earlier, in the 1947 Study, the principle of a three-year life was used as a very general guide.[17] In cases where the available

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[16] *Ibid.*, p. 33.

[17] Jaffe, *op. cit.*, Part 1, p. 17

information indicated that an item with an estimated useful life of more than three years was charged by an industry to current account rather than to capital account, the input-output classification system adhered to the specific industry practices. By contrast, in the 1958 Study, an average useful life of more than one year was used as the criterion in deciding whether or not to classify an item as investment.[18]

Secondly, the scope of capital formation must be defined quite carefully. For example, in the 1947 Study, outlays for plant and equipment included not only new private construction and outlays for new equipment as defined by the U. S. Department of Commerce for national income and product accounting purposes, but also covered many miscellaneous charges to capital account, such as outlays for certain materials and labor charged to capital account (e.g., installation of telephone equipment), receipts of title abstract companies, commissions on transfers of real property, the value of work done in motion picture production, architectural and engineering fees not included in current production costs, research and development work by aircraft companies, and trade margins on sales of second-hand equipment.[19] In addition to new private construction, the cost of additions and alterations and major improvements was considered as part of private capital formation.[20] In the 1958 Study, maintenance and repair construction and major improvements were not considered as capital formation.[21] On the other hand, the net purchase of used equipment and structures, net purchases by individuals of used houses, and real estate commissions earned from the sale of structures (excluding commissions earned from the sale of land) were included in the measurement of private capital formation.[22]

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[18] U.S., Department of Commerce, Office of Business Economics, National Income Division, *loc. cit.*

[19] Ritz and Rudney, *op. cit.,* p. 34.

[20] Jaffe, *loc. cit.*

[21] U.S. Department of Commerce, Office of Business Economics, National Economics Division Staff, "The Transactions Table of the 1958 Input-Output Study, Revised Direct and Total Requirements Data", *Survey of Current Busines,* XLIX, 9 (September, 1965), 39, Table 1, intersection of Row 12-*Maintenance and Repair Construction*-and *gross private fixed capital formation column.*.

[22] U.S. Department of Commerce, Office of Business Economics, National Income Division, *op. cit.,* Appendix 2, p. 4.

Thirdly, the question as to what sectors of the economy to cover must be settled. The central problem here is that of defining the *private* sector, which inevitably involves the dilemma of how to treat nongovernment nonprofit institutions and government industries that have been shifted into the endogenous part of the input-output table. In the impressive explanatory documentation of the 1947 Study, for example, there is hardly any mention of the inclusion or exclusion of nonprofit institutions in the measurement of capital formation. By contrast, it is easy to ascertain that in the 1958 Study, nonprofit institutions were included in the measurement of capital formation.[23] Regarding government industries, the general intention in the 1947 Study was to exclude their capital expenditures from the gross private capital formation column and show them as part of government expenditures. However, absence of adequate data on capital expenditures of government industries resulted in most of such capital expenditures being charged to the business investment account. On the other hand, construction work for government industrial operations could generally be identified and was handled as a purchase by the government sector.[24] In the 1958 Study, this point is left completely obscure. The only available clue on this point is given in a footnote, which states, in effect, that public maintenance and repair construction for water and sewer facilities and for highway toll roads is allocated to government enterprises.[25] Where in the input-output table these entries are shown is not made clear.

4. Net Inventory Change

Net inventory change measures the value of change in the stocks of the products of each sector, regardless of which industry actually holds them. This is different from inventory change *in* each industry, representing the change (i.e., accumulation or depletion on a net basis) in the inventories of each industry. In an input-output table, the net

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[23] *Ibid.*

[24] Jaffe, *loc. cit.*

[25] Frumkin, *loc. cit.*

inventory change column provides the balance between the output of each sector and the total consumption of its products. Current production includes products that are not consumed during the accounting period but end up in inventories. By the same token, consumption may come not only from current production and imports, but also from inventories held by the producer, the consumer, the government (i.e., Commodity Credit Corporation inventories) or by the distributive sectors (e.g., trade companies, warehouses, etc.). To the extent that current consumption comes from inventories, it is not included in current production. Thus, for a given sector, adding inventory increases of the products of that sector to, and subtracting depletions from, the consumption of that sector's products achieves the balance with the total output of the sector.[26]

In the 1947 Study, the measurement of net inventory change covered only finished products.[27] For the most part, inventories were expressed in terms of book value, except for those held by the agriculture, wholesale and retail trade sectors. In these latter cases, an attempt was made to revalue the respective inventories in terms of average 1947 prices.[28]

In the 1958 Input-Output Study, it is not clear whether the measurement of net inventory change covered only finished products or whether it also included raw material and work-in-process inventories. The value of net change in nonfarm inventories was converted from book values to average 1958 prices by means of an inventory valuation adjustment.[29] On the other hand, farm inventories were estimated initially at average prices during the year, and consequently, did not require an inventory valuation adjustment.[30]

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[26] Morris R. Goldman, Martin L. Marimont, and Beatrice N. Vaccara, "The Interindustry Structure of the United States: A Report on the 1958 Input-Output Study," *Survey of Current Business,* XLIV, II (November, 1964), p.17.

[27] Ritz and Rudney, *op. cit.,* p. 35.

[28] *Ibid.* Theoretically, an inventory revaluation should have been carried through for all industries, but because of a variety of difficulties (e.g., developing appropriate price deflators, making appropriate adjustments to industry control totals, and general lack of data), this was not accomplished. Also see Evans and Hoffenberg, *op. cit.,* 108.

[29] This valuation adjustment was made in the total only and not for individual producing sectors. See U.S. Department of Commerce, Office of Business Economics, National Income Division, *loc. cit.*

[30] *Ibid.*

5. Foreign Trade (Gross Exports)

The foreign trade column in an input-output table shows the total outflow of goods and services produced domestically during the accounting period to other countries for purposes of either intermediate or final consumption. The algebraic summation of the imports row total(s) *and* the foreign trade (gross exports) column total yields net exports as recorded in national income and product accounts.[31] As in the rest of the input-output table, the entries in the foreign trade column are expressed in producer's values. The necessary trade margin, transportation and insurance costs incurred in bringing the exports to the port of exit are charged to foreign trade by the relevant domestic distributive sectors.[32]

In the 1947 Study, the territorial boundaries of the domestic economy were defined in terms of the continental United States.[33] Thus, the outflow of goods and services to noncontiguous United States territories and possessions was counted as exports. Conversely, shipments from such areas into the continental United States were considered as imports. In the 1958 Study, it is not immediately clear, at least in the published accounts, how the territorial boundaries of the domestic economy were drawn.

In the 1947 Study, the foreign trade sector included not only the direct outflow of domestically produced goods, but also net unilateral transactions for which there were no tangible compensations.[34] Capital flows, both long and short term, and changes in the gold

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[31] This, however, is by no means automatic, since there usually remain some minor differences between national income and input-output levels of gross exports and gross imports. In the 1958 Input-Output Study of the United States, for example, several items were treated on a gross basis which were shown on a net basis in the existing national income and product accounts and the balance of payments statements (*ibid.,* footnote 1).

[32] Ritz and Rudney, *op. cit.,* p. 31.

[33] *Ibid.*

[34] These unilaterals included mainly government exports for relief purposes. In order to offset the entries shown in the foreign trade column, the government sector was shown as paying a net total to the foreign trade sector (row). Exports by relief organizations and by individuals, insofar as they could be identified, were deleted from the foreign trade column and shown as purchases by the household sector. The household sector, in turn, was shown as purchasing an amount equivalent to such remittances from the foreign trade sector in order to keep financial balance among the various sectors. For these and other details, see Jaffe, *op. cit.,* Part 1, p. 31.

stock were omitted. Also omitted were the exports of used items, except for the distributive charges incurred in selling and transporting them to the port of exit.[35] Remittances by U. S. citizens to foreign countries, balanced against remittances by foreigners to the United States, appeared in the table as a payment by the household sector to the foreign trade sector.[36]

In the 1958 Study, included in the foreign trade column were such items as re-exports,[37] private remittances in kind,[38] government nonmilitary grants in kind, and expenditures of foreigners traveling or temporarily residing in the United States.[39] Government receipts of interest from foreigners were also shown as an export, while government payments of interest to foreigners were shown as a service import.[40]

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[35] Ritz and Rudney, *loc. cit.*

[36] It will be recalled that expenditures by U. S. citizens abroad were shown similarly on a net basis as a purchase by the household sector from the foreign trade (imports) row. The foreign trade-to-household *cell* also included net profits of the branch offices abroad of domestic firms, which were shown as payments by the foreign trade sector (row) to the household sector (column). See Jaffe, *loc. cit.*

[37] The total value of re-exports is included in the *cell* where the noncompetitive imports row (Row 80A) intersects the *net exports* column. Since this entry is an aggregate net total, the value of re-exports is not separately shown. See, U.S. Department of Commerce, Office of Business Economics, National Income Division, *op. cit.,* Appendix 2, p. 3.

[38] Private (personal and institutional) remittances in cash and in kind were taken out of personal consumption expenditures (PCE) and shown as a negative *payment* by the household sector to an account (Row 85) entitled "Rest of the World". Treated in this manner, such remittances were excluded from imports (*ibid.,* Appendix 2, p. 1).

[39] It will be recalled that in the 1947 Interindustry Relations Study, the difference between the personal consumption expenditures of U.S. citizens abroad *and* the expenditures of foreigners in the United States was treated as a negative purchase by the household sector from the noncompetitive imports row.

[40] U.S. Department of Commerce, Office of Business Economics, National Income Division, *op. cit.,* Appendix 2, p. 3.

C. DEFINITION AND MEASUREMENT OF PRIMARY INPUTS (VALUE ADDED)

Total inputs of an industry consist not only of intermediate materials and supplies but also of primary inputs that represent charges against final product. These charges against final product are collectively called *value added* and are recorded in the autonomous row(s) of an input-output table. If the input-output table is fully integrated with national income and product accounts, *value added* consists of the following components:[41]

(a) Compensation of employees, which is the income accruing to persons in an employee status as renumeration for their work. It is the sum of *wages and salaries* (i.e., the monetary remuneration of employees, inclusive of executives' compensation, commissions, tips, and bonuses, and of payments in kind which represent income to the recipients) *and supplements to wages and salaries* (i.e., employer contributions for social insurance *plus* other labor income). Employer contributions for social insurance covers employer payments under social security, federal and state unemployment insurance, railroad retirement and unemployment insurance, government retirement and a few other social insurance programs that are less significant. Other labor income covers employer contributions to private pension plans, health, unemployment and welfare funds, compensation for injuries, directors' fees, and a few other items.

(b) Tax and nontax payments to all levels of government (including corporate income tax, social security, excise, property, and license taxes, and other payments, such as for crop insurance, charges for water service, tolls, etc.) *plus* the current surplus of government enterprises *minus* government subsidies;

(c) Capital consumption allowances, which consist of depreciation charges, accidental damage to fixed business capital, and capital outlays charged to current expenses;

(d) Other charges against final product, which is the sum of the following specific items:

- i Undistributed corporate profits and/or proprietors' income
- ii Payments on dividends
- iii Net interest payments
- iv Business transfer payments, including bad debts, contributions, prizes, and gifts
- v Payments of rent other than to persons primarily engaged in the real estate business, and payments for royalties, patents, copyrights, and rights to natural resources
- vi Other deductions, which cover a variety of minor items and adjustments.

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[41] See the following sources: Jaffe, *op. cit.,* Part 1, pp. 16-24; Gardner Ackley, *Macroeconomic Theory* (New York: The Macmillan Co., 1961), pp. 38-77; U. S. Department of Commerce, Office of Business Economics, *The National Income and Product Accounts of the United States, 1929-1965, Statistical Tables; A Supplement to the Survey of Current Business* (Washington, D.C.: U. S. Government Printing Office, 1966), pp. ix-xi, and Tables 1.9, 1.10, 2.1, 6.1, 6.2, and 6.7.

Alternatively (and much more simply), *value added* can be defined as total value of production (f.o.b. plant) *minus* the cost of materials and supplies purchased from other firms. Here, total value of production is equal to total sales adjusted for changes in inventories, wherever held. In concept, input-output definition of *value added* is closely approximated by the *value added* series of U. S. Bureau of the Census. Another, but somewhat less perfect, approximation is provided by the *national income originating* concept used by the U. S. Office of Business Economics.

Under the Census concept, value added (by manufacture) is derived by subtracting the total cost of materials (including materials, supplies, fuel, electric energy, cost of resales and miscellaneous receipts) *from* the value of shipments (including resales) and other receipts, and by adjusting the resulting amount by the net change in finished products and work-in-process inventories between the beginning and end of the year.[42]

National income originating in each industry, as defined by U. S. Office of Business Economics, is the sum of factor costs incurred by the industry in production, and represents the net value added to production by the industry. It is, therefore, a more *net* concept of value added than used by U. S. Bureau of the Census.[43] *Income originating* excludes, in addition to the cost of materials, such other costs as depreciation charges, state and local taxes (other than corporate income taxes), allowance for bad debts, and purchases of services from nonmanufacturing enterprises (e.g., contract costs involved in maintenance and repair, services of development and research firms, services of engineering and management consultants, advertising, telephone and telegraph expense, insurance, royalties, patent fees, etc.).[44] In recent years, *national income originating* in the manufacturing industries as a whole has comprised approximately 75 to 77 percent of value added in manufacturing as defined by the U. S. Bureau of the Census.[45]

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[42] U.S. Bureau of the Census, *1963 Census of Manufactures,* Vol. II, *Industry Statistics,* Part I, *Major Groups 20-28* (Washington, D.C.: U. S. Government Printing Office, 1966), p. 22.

[43] U. S. Department of Commerce, Office of Business Economics, *U.S. Income and Output; A Supplement to the Survey of Current Business* (Washington, D.C.: U.S. Government Printing Office, 1958), Table I-10, footnote 1.

[44] U. S. Bureau of the Census, *op. cit.,* p. 23.

[45] *Ibid.,* p. 22

In the 1947 Study, the *household* and *government* components of primary inputs (value added) were shown separately, and the rest of the charges against final product (including capital consumption allowances) were aggregated and shown in another row.[46] In the 1958 Study, only value added as a whole was shown in a single row.[47]

In principle, the household (i.e., compensation of employees) component of primary inputs can be expressed in terms of labor required by different skill or occupational categories, where labor can be expressed in terms of number of employees or total man-hours. Such an *occupation-by-industry* table can be converted into a direct labor requirements matrix, in which each element may express direct labor input requirements of type i per unit of industry j's output. Such an extension of the input-output framework can be quite fruitful, for example, in analyzing the effects of disarmament, changes in consumer preferences, shifts in government expenditures from defense to nondefense needs, etc., on industrial employment levels.

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[46] U. S. Department of Labor, Bureau of Labor Statistics, *Table I — Interindustry Flow of Goods and Services by Industry of Origin and Destination: Continental United States, 1947* (October, 1952).

[47] See, for instance, U.S. Department of Commerce, Office of Business Economics, National Economics Division Staff, *op. cit.,* Table 1.

APPENDIX D
MATHEMATICAL DIGRESSION ON THE INVERSION OF PARTITIONED MATRICES

The purpose of this appendix is to provide the mathematical background that is necessary for understanding and checking the results given in Chapters II, IV, and V.

Let L be a nonsingular partitioned matrix,

$$L = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{1}$$

where α, β, γ, and δ are four nxn submatrices. Further, let

$$L^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^{-1} = \begin{bmatrix} R & S \\ T & U \end{bmatrix} \tag{2}$$

where R, S, T, and U represent four nxn submatrices in the inverse of the L– matrix, suci that

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} R & S \\ T & U \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} . \tag{3}$$

From the property $LL^{-1} = L^{-1}L = I$, we can obtain four equations for the four unknown submatrices R, S, T, U:

$$\text{(4a)} \qquad \alpha R + \beta T = I$$

$$\text{(4b)} \qquad \alpha S + \beta U = 0$$

$$\text{(4c)} \qquad \gamma R + \delta T = 0$$

$$\text{(4d)} \qquad \gamma S + \delta U = I$$

which can be written more compactly as

$$\text{(5)} \qquad \begin{bmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha & 0 & \beta \\ \gamma & 0 & \delta & 0 \\ 0 & \gamma & 0 & \delta \end{bmatrix} \begin{pmatrix} R \\ S \\ T \\ U \end{pmatrix} = \begin{pmatrix} I \\ 0 \\ 0 \\ I \end{pmatrix}$$

or as

$$\text{(6)} \qquad \begin{pmatrix} R \\ S \\ T \\ U \end{pmatrix} = \begin{bmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha & 0 & \beta \\ \gamma & 0 & \delta & 0 \\ 0 & \gamma & 0 & \delta \end{bmatrix}^{-1} \begin{pmatrix} I \\ 0 \\ 0 \\ I \end{pmatrix}$$

The determinant of the parameter matrix, which can be denoted as $|P|$, is first computed, and we have

$$\text{(7)} \qquad |P| = \alpha^2 \delta^2 + \gamma\beta^2\gamma - 2\alpha\gamma\delta\beta.$$

Then, dividing each element of the adjoint of the parameter matrix we obtain the inverse of the parameter matrix, P^{-1}, as follows:

(8)

$$P^{-1} = \begin{bmatrix} \frac{\alpha\,\delta^2 - \gamma\,\delta\,\beta}{|P|} & 0 & \frac{\beta^2\,\gamma - \delta\,\alpha\,\beta}{|P|} & 0 \\ 0 & \frac{\alpha\,\delta^2 - \delta\,\gamma\,\beta}{|P|} & 0 & \frac{\beta^2\,\gamma - \delta\,\beta\,\alpha}{|P|} \\ \frac{\beta\gamma^2 - \delta\,\gamma\,\alpha}{|P|} & 0 & \frac{\alpha^2\,\delta - \gamma\,\beta\,\alpha}{|P|} & 0 \\ 0 & \frac{\beta\,\gamma\,\alpha + \beta\,\gamma^2}{|P|} & 0 & \frac{\alpha^2\,\delta - \gamma\,\alpha\,\beta}{|P|} \end{bmatrix}$$

Then, post-multiplying P^{-1} by the column vector (I 0 0 I)′, we obtain R, S, T, and U as follows:

(9a) $$R = (\alpha\,\delta^2 - \gamma\,\delta\,\beta)\;(\alpha^2\,\delta^2 + \gamma\,\beta^2\,\gamma - 2\,\alpha\,\gamma\,\delta\,\beta)^{-1}$$

(9b) $$S = (\beta^2\,\gamma - \delta\,\beta\,\alpha)\;(\alpha^2\,\delta^2 + \gamma\,\beta^2\,\gamma - 2\,\alpha\,\gamma\,\delta\,\beta)^{-1}$$

(9c) $$T = (\beta\gamma^2 - \delta\,\gamma\,\alpha)\;(\alpha^2\,\delta^2 + \gamma\,\beta^2\,\gamma - 2\,\alpha\,\gamma\,\delta\,\beta)^{-1}$$

(9d) $$U = (\alpha^2\,\delta - \gamma\,\alpha\,\beta)\;(\alpha^2\,\delta^2 + \gamma\,\beta^2\,\gamma - 2\,\alpha\,\gamma\,\delta\,\beta)^{-1}.$$

Alternatively, we can solve for the four unknowns R, S, T, and U in (4c) somewhat more easily as follows:[1]

(10) $$T = \delta^{-1}\,\gamma\,R.$$

Substituting this into (4a), we obtain

(11) $$\alpha\,R - \beta\,\delta^{-1}\,\gamma\,R = I$$

which can be written as

(12) $$(\alpha - \beta\,\delta^{-1}\,\gamma)\;R = I.$$

[1] Refer to G. Hadley, *Linear Algebra* (Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1961), pp. 107-111.

and solve for R:

(13) $$R = (\alpha - \beta\,\delta^{-1}\,\gamma)^{-1}.$$

Now, using (4d), we arrive at

(14) $$U = \delta^{-1} - \delta^{-1}\,\gamma\,S.$$

From (4b) and (14), we have

(15) $$\alpha\,S + \beta\,(\delta^{-1} - \delta^{-1}\,\gamma\,S) = 0$$

(16) $$\alpha\,S + \beta\,\delta^{-1} - \beta\,\delta^{-1}\,\gamma\,S = 0$$

(17) $$(\alpha - \beta\,\delta^{-1}\,\gamma)\,S = -\beta\,\delta^{-1}.$$

From (13), we get

(18) $$S = -R\,\beta\,\delta^{-1}.$$

We have obtained four formulas (13), (18), (10), and (14) which can be solved sequentially for R, S, T, and U. They are:

(19a) $$R = (\alpha - \beta\,\delta^{-1}\,\gamma)^{-1},$$

(19b) $$S = -R\,\beta\,\delta^{-1},$$

(19c) $$T = -\delta^{-1}\,\gamma\,R,$$

(19d) $$U = \delta^{-1} - \delta^{-1}\,\gamma\,S.$$

Since L^{-1} exists, the submatrices R, S, T, U exist. Hence, if δ^{-1} exists, all the operations can be carried out, and R, S, T, U can be computed. In summary, we have:

(20a) $$R = (\alpha - \beta\,\delta^{-1}\gamma)^{-1},$$

(20b) $$S = -(\alpha - \beta\,\delta^{-1}\gamma)^{-1}\,\beta\,\delta^{-1},$$

(20c) $$T = -\,\delta^{-1}\gamma\,(\alpha - \beta\,\delta^{-1}\,\gamma)^{-1},$$

(20d) $$U = \delta^{-1} - \delta^{-1}\gamma\,[\,-(\alpha - \beta\,\delta^{-1}\gamma)^{-1}\,\beta\,\delta^{-1}\,],$$

where (20a) – (20d) are equivalent to the results given in (9a – 9d) and perhaps easier to follow.

The results obtained in Eqs. (19a) through (19d) or in Eqs. (20a) through (20d) are applicable in finding the inverse of larger partitioned matrices. Let us assume, for example, that

(21) $$L = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

and

(22) $$L^{-1} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

where, as before, each submatrix is square and is of the order nxn. We can further partition both L and L^{-1}, such that we have

(23)
$$\alpha = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \beta = \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix}, \gamma = (A_{31} \quad A_{32}), \text{ and } \delta = A_{33};$$

(24)
$$R = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, S = \begin{pmatrix} B_{13} \\ B_{23} \end{pmatrix}, T = (B_{31} \quad B_{32}), \text{ and } U = B_{33}.$$

First, we can easily compute R as follows:

$$R = (\alpha - \beta\delta^{-1}\gamma)^{-1}$$

(25)
$$= \left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix} [A_{33}]^{-1} \quad (A_{31} \quad A_{32}) \right\}^{-1}$$

$$= \begin{bmatrix} A_{11} - A_{13} A_{33}^{-1} A_{31} & A_{12} - A_{13} A_{33}^{-1} A_{32} \\ A_{21} - A_{23} A_{33}^{-1} A_{31} & A_{22} - A_{23} A_{33}^{-1} A_{32} \end{bmatrix}^{-1}.$$

Again, using earlier results, we see that

(26a)
$$B_{11} = \Big\{ (A_{11} - A_{13} A_{33}^{-1} A_{31}) - (A_{12} - A_{13} A_{33}^{-1} A_{32}) (A_{22} - A_{23} A_{33}^{-1} A_{32})^{-1} (A_{21} - A_{23} A_{33}^{-1} A_{31}) \Big\}^{-1}$$

(26b) $B_{12} = -B_{11}\,(A_{12} - A_{11}\,A_{33}^{-1}\,A_{32})\,(A_{22} - A_{23}\,A_{33}^{-1}\,A_{32})^{-1}$

(26c) $B_{21} = -(A_{22} - A_{23}\,A_{33}^{-1}\,A_{32})^{-1}\;(A_{21} - A_{23}\,A_{33}^{-1}\,A_{31})\;B_{11}$

(26d) $B_{22} = (A_{22} - A_{23}\,A_{33}^{-1}\,A_{32})^{-1} - (A_{22} - A_{23}\,A_{33}^{-1}\,A_{32})^{-1}\,(A_{21} - A_{23}\,A_{33}^{-1}A_{31})\,B_{12}.$

Secondly, for **S** (i.e., B_{13} and B_{23}) we have

$$S = \;-\;R\beta\delta^{-1}$$

(27)
$$\begin{pmatrix} B_{13} \\ B_{23} \end{pmatrix} = \begin{bmatrix} -\,B_{11} & -\,B_{12} \\ -\,B_{21} & -\,B_{22} \end{bmatrix} \begin{pmatrix} A_{13} \\ A_{23} \end{pmatrix} [A_{33}]^{-1}\,.$$

$$= \begin{bmatrix} -(B_{11}\;A_{13}\;A_{33} + B_{12}\;A_{23}\;A_{33}) \\ -(B_{21}\;A_{13}\;A_{33} + B_{22}\;A_{23}\;A_{33}) \end{bmatrix}.$$

Thirdly, we can obtain T (i.e., B_{31} and B_{32}) as follows:

$$T = -\delta^{-1}\,\gamma\,R$$

$$(B_{31}\;B_{32}) = -\;A_{33}^{-1}\;(A_{31}\quad A_{32}) \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

(28)
$$= \Big\{-\;(A_{33}^{-1}\;A_{31}\;B_{11} + A_{33}^{-1}A_{32}\;B_{21})$$

$$-\;(A_{33}^{-1}\;A_{31}\;B_{12} + A_{33}^{-1}\;A_{32}\;B_{22})\Big\}.$$

Finally, for U we have

$$U = \delta^{-1} - \delta^{-1} \gamma S$$

$$= A_{33}^{-1} - A_{33}^{-1} (A_{31} \quad A_{32}) \begin{pmatrix} B_{13} \\ B_{23} \end{pmatrix}$$

(29) $$= A_{33}^{-1} - (A_{33}^{-1} A_{31} \quad A_{33}^{-1} A_{32}) \begin{pmatrix} B_{13} \\ B_{23} \end{pmatrix}$$

$$= A_{33}^{-1} - (A_{33}^{-1} A_{31} B_{13} + A_{33}^{-1} A_{32} B_{23})$$

$$= A_{33}^{-1} (I - A_{31} B_{13} - A_{32} B_{23}).$$

APPENDIX E

TABLE E-1

THE MAIN COMPUTER PROGRAM USED IN THE EXPERIMENTS FOR THE "INDUSTRY TECHNOLOGY" MODELS

(NOTE: This program refers specifically to the 17x17-order "industry technology" system. Following the read statements, the dimensions must be changed into 45, 60, 79, in order to make this program workable at other levels of sectoral aggregation.)

```
FORTRAN IV G LEVEL 1, MOD 3            MAIN             DATE = 69135        05/59/36              PAGE 0001

 0001              DIMENSION LLM(79),X(79,79)
 0002              DIMENSION B1(79),B2(79),B3(79),B4(79),B5(79),B6(79),B7(79),B8(79),
                  1B9(79),B10(79),B11(79),B12(79),B13(79),B14(79),         B17(79),B18
                  2(79),B19(79),B20(79),B21(79),B7A(79),                Z(79),ZC(79)
 0003              DIMENSION B22(79),B23(79),B24(79),B25(79),B29(79)
 0004              DIMENSION P(79),Q(79),SB(2),F(79)
 0005              DIMENSICN AG(17,17),QG(17),UG(17,17),ANS(17,17),AX(79,17)
 0006              DIMENSICN A(79,79),U(79,79),              B26(79)
 0007              COMMON A,MLP
 0008              MLP=6
 0009              IT1=4
 0010              L=1
           C
           C
 0011              N=79
           C
           C
 0012              DO 1 I=1,N
 0013            1 LLM(I)=I
           C             PUT TAPE IN CORF
 0014              DO 2 I=1,N
 0015            2 READ(IT1)(A(I,J),J=1,N)
 0016              CALL MAGG17(A,AG,P,QG)
 0017            3 FORMAT('1A+M        ** KB **')
 0018              WRITE(MLP,3)
 0019              CALL MAT17(LLM,AG,1.0)
           C             CHANGE FLOW TO A+M
 0020              READ(IT1)(Z(I),I=1,N)
 0021              CALL VAGG17(Z,ZC)
 0022          300 FORMAT('1CONTROL TOTALS      ** KB **',//)
 0023              WRITE(MLP,300)
 0024          301 FORMAT(1X,I2,F13.0)
 0025              DO 302 I=1,17
 0026          302 WRITE(MLP,301)I,ZC(I)
           C             DIVIDE FLOW BY SUMS
           C
           C
 0027              N=17
           C
           C
 0028              DO 4 I=1,N
 0029              DO 4 J=1,N
 0030              Z(I)=ZC(I)
 0031            4 AG(J,I)=AG(J,I)/Z(I)
 0032            5 FORMAT('1COEFFICIENT MATRIX      ** KB **')
 0033              WRITE(MLP,5)
 0034              CALL MAT17(LLM,AG,0.0)
           C
```

```
C
      N=79
C
C
      READ(IT1)(B6(I),I=1,N)
      READ(IT1)(B1(I),I=1,N)
      READ(IT1)(B2(I),I=1,N)
      READ(IT1)(B3(I),I=1,N)
      READ(IT1)(B4(I),I=1,N)
      READ(IT1)(B5(I),I=1,N)
      READ(IT1)(B25(I),I=1,N)
      READ(IT1)(B21(I),I=1,N)
      READ(IT1)(B22(I),I=1,N)
      READ(IT1)(B23(I),I=1,N)
      READ(IT1)(B7(I),I=1,N)
      READ(IT1)(B7A(I),I=1,N)
      READ(IT1)(B8(I),I=1,N)
      B8(30)=64.2
      B8(45)=2030.2
      READ(IT1)(B9(I),I=1,N)
      READ(IT1)(B10(I),I=1,N)
      B10(8)=10680.0
      REWIND IT1
      CALL VAGG17(B6,B6)
      CALL VAGG17(B1,B1)
      CALL VAGG17(B2,B2)
      CALL VAGG17(B3,B3)
      CALL VAGG17(B4,B4)
      CALL VAGG17(B5,B5)
      CALL VAGG17(B7,B7)
      CALL VAGG17(B7A,B7A)
      CALL VAGG17(B8,B8)
      CALL VAGG17(B9,B9)
      CALL VAGG17(B10,B10)
      CALL VAGG17(B21,B21)
      CALL VAGG17(B22,B22)
      CALL VAGG17(B23,B23)
      CALL VAGG17(B25,B25)
C           READ IN U
      DO 61 I=1,79
      DO 61 J=1,79
   61 U(I,J)=0.0
   62 FORMAT(1X,I2,1X,I2,2X,F9.0)
   63 READ(3,62)II,JJ,U(II,JJ)
      IF(II-79)63,64,63
   64 IF(JJ-68)63,65,65
   65 CONTINUE
      Z( 1)=23964877.
```

```
Z( 2)=20669423.
Z( 3)=  905486.
Z( 4)= 1012771.
Z( 5)=  746480.
Z( 6)=  987894.
Z( 7)= 2729914.
Z( 8)= 9133814.
Z( 9)= 1351992.
Z(10)=  308430.
Z(11)=52416000.
Z(12)=16867670.
Z(13)= 2301017.
Z(14)=60620483.
Z(15)= 5767103.
Z(16)= 9932830.
Z(17)= 1853025.
Z(18)=14032219.
Z(19)= 1730234.
Z(20)= 7429073.
Z(21)=  376967.
Z(22)= 3053023.
Z(23)= 1246093.
Z(24)= 8701023.
Z(25)= 3436165.
Z(26)= 5965364.
Z(27)= 9348472.
Z(28)= 3406441.
Z(29)= 5779196.
Z(30)= 1702477.
Z(31)=16013897.
Z(32)= 6062644.
Z(33)=  862180.
Z(34)= 2959548.
Z(35)= 2087515.
Z(36)= 6945233.
Z(37)=18152783.
Z(38)= 8336500.
Z(39)= 2005733.
Z(40)= 6763570.
Z(41)= 2899741.
Z(42)= 4978194.
Z(43)= 1647910.
Z(44)= 2047177.
Z(45)= 2523434.
Z(46)=  777466.
Z(47)= 2746435.
Z(48)= 1961634.
Z(49)= 2868055.
```

```
0127          Z(50)= 1326907.
0128          Z(51)= 1553671.
0129          Z(52)= 1660906.
0130          Z(53)= 3983555.
0131          Z(54)= 2902910.
0132          Z(55)= 1883359.
0133          Z(56)= 4720699.
0134          Z(57)= 2076976.
0135          Z(58)= 1176380.
0136          Z(59)=21385585.
0137          Z(60)= 9580256.
0138          Z(61)= 3322025.
0139          Z(62)= 2558465.
0140          Z(63)= 1293509.
0141          Z(64)= 4223820.
0142          Z(65)=29502962.
0143          Z(66)= 8886570.
0144          Z(67)=    16602.
0145          Z(68)=17152507.
0146          Z(69)=90945115.
0147          Z(70)=25632465.
0148          Z(71)=55274311.
0149          Z(72)=10848678.
0150          Z(73)=15879226.
0151          Z(74)=   534270.
0152          Z(75)= 7822002.
0153          Z(76)= 5370848.
0154          Z(77)=22103563.
0155          Z(78)= 3144265.
0156          Z(79)=   741900.
0157          DO 650 I=1,79
0158      650 U(I,I)=Z(I)
0159          CALL MAGG17(U,UG,P,QG)
        C
        C
0160          N=17
        C
        C
0161          DO 68 J=1,N
0162          SUM=0.0
0163          DO 66 I=1,N
0164       66 SUM=SUM+UG(I,J)
0165          DO 67 I=1,N
0166       67 UG(I,J)=UG(I,J)/SUM
0167       68 CONTINUE
0168     9901 FORMAT('1U')
0169          WRITE(MLP,9901)
0170          CALL MAT17(LLM,UG,0.0)
```

```
      DO 8 I=1,N
      DO 8 J=1,N
    8 ANS(I,J)=0.0
      DO 9 K=1,N
      DO 9 J=1,N
      DO 9 I=1,N
    9 ANS(K,J)=AG(K,I)*UG(I,J)+ANS(K,J)
   10 FORMAT('1(A+M)U       ** KB **')
      WRITE(MLP,10)
      CALL MAT17(LLM,ANS,0.0)
      DO 11 I=1,N
      DO 11 J=1,N
   11 ANS(I,J)=-ANS(I,J)
      DO 12 I=1,N
   12 ANS(I,I)=1.0+ANS(I,I)
   13 FORMAT('1I-(A+M)U       ** KB **')
      WRITE(MLP,13)
      CALL MAT17(LLM,ANS,0.0)
      CALL MINV(ANS,N,DLL,P,Q)
   14 FORMAT('1A INVERSE       ** KB **')
      WRITE(MLP,14)
   15 FORMAT('0',10X,'DETERMINANT',2X,F14.5)
      WRITE(MLP,15)DLL
      CALL MAT17(LLM,ANS,0.0)
      DO 16 I=1,N
      B17(I)=0.0
   16 B18(I)=0.0
      DO 17 I=1,N
      DO 17 J=1,N
   17 B17(I)=ANS(I,J)*B9(J)+B17(I)
      DO 18 I=1,N
   18 B18(I)=B17(I)-B8(I)
      DO 19 I=1,N
   19 B26(I)=B10(I)-B8(I)
   20 FORMAT(1X,I2,10X,F14.5,10X,F14.5,10X,F14.5)
   21 FORMAT ('1OUTPUT LEVELS 1961',5X,I2,'X1 VECTORS',//)
   22 FORMAT('0',17X,'PREDICTED',16X,'ACTUAL',15X,'PRED - ACT',/)
  230 DO 23 I=1,N
   23 B29(I)=B17(I)-B10(I)
      WRITE(MLP,21)N
      WRITE(MLP,24)
   24 FORMAT('0X-CAP D SUB T',/)
      WRITE(MLP,22)
      WRITE(MLP,20) (LLM(I),B17(I),B10(I),B29(I),I=1,N)
      CALL ERROF(B17,B10,N)
      DO 25 I=1,N
   25 B29(I)=B18(I)-B26(I)
      WRITE(MLP,21)N
```

```
0219              WRITE(MLP,26)
0220           26 FORMAT('CX-CAP D SQ SUB T',/)
0221              WRITE(MLP,22)
0222              WRITE(MLP,20) (LLM(I),B18(I),B26(I),B29(I),I=1,N)
0223              CALL ERROF(B18,B26,N)
0224              IF(L-2)265,999,999
0225          265 CONTINUE
        C
        C
0226              N=79
        C
        C
0227              DO 27 J=1,95
0228           27 READ(IT1)(Z(I),I=1,N)
0229              DO 28 I=1,N
0230           28 READ(IT1)(A(I,J),J=1,N)
0231              A(3,33)=934.0
0232              A(3,34)=0.0
0233              A(2,64)=7780.0
0234              A(26,69)=1203098.0
0235              A(26,76)=118766.C
0236              A(25,76)=788.0
0237              A( 1,78)=1737.0
0238              A( 1,77)=4552.0
0239              A( 1,76)=10865.0
0240              DO 281 J=1,67
0241              A(73,J)=A(73,J)+A(67,J)
0242          281 A(67,J)=0.0
0243              DO 282 J=68,72
0244              A(73,J)=A(73,J)+A(67,J)
0245          282 A(67,J)=0.0
0246              DO 283 J=74,79
0247              A(73,J)=A(73,J)+A(67,J)
0248          283 A(67,J)=0.0
0249           29 FORMAT('1A+M      ** NPA **')
0250              WRITE(MLP,29)
0251              CALL MAGG17(A,AG,P,QG)
0252              CALL MAT17(LLM,AG,1.0)
0253              READ(IT1)(Z(I),I=1,79)
        C                   BETTER VALUES FOR **NPA** CONTROL TOTALS
0254              Z( 1)=23842363.0
0255              Z( 2)=20735344.0
02J6              Z(12)=16875C00.0
0257              Z(14)=63151574.0
0258              Z(59)=21846C52.0
0259              Z(65)=32801002.0
0260              Z(68)=2C194431.0
0261              Z(69)=94350108.0
```

```
0262              Z(70)=25695465.0
0263              Z(71)=61934295.0
0264              Z(73)=16962314.0
0265              Z(77)=22677963.0
0266              CALL VAGG17(Z,Z)
0267          290 FORMAT('1CONTROL TOTALS      ** NPA **',//)
0268              WRITE(MLP,290)
            C
            C
0269              N=17
            C
            C
0270              DO 291 I=1,N
0271          291 WRITE(MLP,301)I,Z(I)
0272              REWIND IT1
0273              DO 30 J=1,N
0274              DO 30 I=1,N
0275           30 AG(I,J)=AG(I,J)/Z(J)
0276           31 FORMAT('1COEFFICIENT MATRIX      ** NPA **')
0277              WRITE(MLP,31)
0278              CALL MAT17(LLM,AG,0.0)
0279              DO 32 I=1,N
0280              DO 32 J=1,N
0281           32 AG(I,J)=-AG(I,J)
0282              DO 33 I=1,N
0283           33 AG(I,I)=AG(I,I)+1.0
0284           34 FORMAT('1I-(A+M)      ** NPA **')
0285              WRITE(MLP,34)
0286              CALL MAT17(LLM,AG,0.0)
0287              CALL MINV(AG,N,DLL,P,Q)
0288           35 FORMAT('1A INVERSE      ** NPA **')
0289              WRITE(MLP,35)
0290           36 FORMAT('0',10X,'DETERMINANT',2X,F14.5)
0291              WRITE (MLP,36)DLL
0292              CALL MAT17(LLM,AG,0.0)
0293              DO 37 I=1,N
0294              B17(I)=0.0
0295           37 B18(I)=0.0
0296              DO 38 I=1,N
0297              DO 38 J=1,N
0298           38 B17(I)=AG(I,J)*B9(J)+B17(I)
0299              DO 39 I=1,N
0300           39 B18(I)=B17(I)-B8(I)
0301              L=2
0302              GO TO 230
0303          999 CALL EXIT
0304              END
```

```
0001            SUBROUTINE ERROF(ZC,Z,N)
0002            DIMENSION E(79),TH(79),ZC(79),X(2,79),Z(79),A(79,79)
               1,Z1(17),ZC1(17),Z2(17),ZC2(17)
0003            COMMON A,MLP
          C  **    STEP 1A
0004            KKK=N
0005         17 DO 1 I=1,N
0006          1 E(I)=ABS(ZC(I)-Z(I))/Z(I)
          C  **    STEP 1B
0007            E1=0.
0008            DO 2 I=1,N
0009       2    E1=E(I)+E1
0010            E1=E1/FLOAT(N)
          C  **    STEP 2
          C  **    COMPUTE MEAN SQUARE PREDICTION ERROR (UNADJUSTED
0011            S2=0.
0012            DO 3 I=1,N
0013          3 S2=(ZC(I)-Z(I))**2+S2
0014            S2=S2/FLOAT(N)
          C  **    PREDICTION STANDARD DEVIATION
0015            S=SQRT(S2)
          C  **    STANDARD ERROR OF PREDICTION
0016            SS=SQRT(S2/FLOAT(N))
          C  **    KURTOSIS  PEAKEDNESSS  OF PREDICTION ERRORS
0017            FK=0.0
0018            DO 4 I=1,N
0019       4    FK=(ZC(I)-Z(I))**4+FK
0020            FK=FK/(FLOAT(N)*S2**2)
          C  **    WEIGHTED MEAN SQUARE PREDICTION ERROR
0021            T1=0.
0022            T2=0.
0023            DO 5 I=1,N
0024            T1=((ZC(I)-Z(I))**2)*Z(I)+T1
0025       5    T2=Z(I)+T2
0026            SC2=T1/T2
          C  **    PREDICTION STANDARD DEVIATION  WEIGHTED
0027            SC2=ABS(SC2)
0028            SC=SQRT(SC2)
          C  **    STANDARD ERROR OF PREDICTION   WEIGHTED
0029            SS2=SQRT(SC2/FLOAT(N))
0030            SK=0.
0031            DO 6 I=1,N
0032       6    SK=(ZC(I)-Z(I))**4+SK
0033            SK=SK/(FLOAT(N)*SC2**2)
          C  **    LN PREDICTION ERRORS-UNWEIGHTED-
          C  **    LN PREDICTION ERRORS
0034            DO 7 I=1,N
0035            IF(ZC(I))72,71,72
```

```
 71   TH(I)=0.0
      GO TO 7
 72   TH(I)=ALOG(ABS(ZC(I)/Z(I)))
  7   CONTINUE
      THA=0.
      DO 8 I=1,N
  8   THA=TH(I)+THA
      THA=THA/FLOAT(N)
      THA=ABS(THA)
C  **     AVERAGE WEIGHTED LN PREDICTION ERROR
      T1=0.
      T2=0.
      DO 9 I=1,N
      T1=TH(I)*Z(I)+T1
  9   T2=Z(I)+T2
      TS=ABS(T1/T2)
      WRITE(MLP,12)
 12   FORMAT('1',9X,'GOODNESS OF FIT TESTS',//)
      WRITE(MLP,10)E1,S2,S,SS,FK,SC2,SC,SS2
 10   FORMAT(' AVERAGE PREDICTION ERROR>',F16.5,/,' MEAN SQUARE PREDICTI
     1ON ERROR UNADJUSTED >',F16.5,/,' PREDICTION STANDARD DEVIATION>',F
     216.5,/,' STANDARD ERROR OF PREDICTION>',F16.5,/,' KURTOSIS PEAKEDN
     3ESS OF PREDICTION ERRORS>',F16.5,/,' WEIGHTED MEAN SQUARE PREDICTI
     4ON ERROR>',F16.5,/,' PREDICTION STANDARD DEVIATION WEIGHTED >',F16
     5.5,/,' STANDARD ERROR OF PREDICTION WEIGHTED >',F16.5)
      WRITE(MLP,11)SK,THA,TS
 11   FORMAT(' KURTOSIS PEAKEDNESS  OF PREDICTION ERROR>',F16.5,/,' AVER
     1AGE UNWEIGHTED LOGARITHMIC PREDICTION ERROR>',F16.5,/,' AVERAGE WE
     2IGHTED LOGARITHMIC PREDICTION ERROR>',F16.5)
      WRITE(MLP,13)
 13   FORMAT('1',9X,'E(I)',15X,'LOG PREDICTION ERROR')
      DO 14 I=1,N
      TH(I)=ABS(TH(I))
 14   WRITE(MLP,15)E(I),TH(I)
 15   FORMAT(F16.5,4X,F16.5)
      RETURN
      END
```

When N>17, both actual and predicted vectors are aggregated and this error routine is repeated.

For example, when N=60, the following instructions are included immediately after instruction 15.

```
      IF(N-17)99,99,16
   16 CONTINUE
      CALL ARG60(Z,Z2)
      CALL ARG60(ZC,ZC2)
      DO 19 I=1,17
      Z1(I)=Z(I)
      ZC1(I)=ZC(I)
      Z(I)=Z2(I)
   19 ZC(I)=ZC2(I)
      N=17
   20 FORMAT(1X,I2,10X,F14.5,10X,F14.5,10X,F14.5)
   21 FORMAT('0    AGGREGATED')
      WRITE(MLP,21)

   22 FORMAT('0',17X,'PREDICTED',16X,'ACTUAL',15X,'PRED - ACT',/)
      WRITE(MLP,22)
      DO 18 I=1,17
      E(I)=ZC(I)-Z(I)
   18 WRITE(MLP,20)I,ZC(I),Z(I),E(I)
      GO TO 17
   99 N=KKK
      DO 23 I=1,17
      Z(I)=Z1(I)
   23 ZC(I)=ZC1(I)
      RETURN
      END
```

```
FORTRAN IV G LEVEL 1, MOD 3          MAGG17          DATE = 69135          05/59/36          PAGE 0001

      SUBROUTINE MAGG17(A,AG,P,QG)
      DIMENSION A(79,79),AG(17,17),P(79),QG(17),AX(79,17)
      DO 1 I=1,79
      DO 1 J=1,79
    1 AG(I,J)=0.0
      DO 4 K=1,79
      DO 2 J=1,79
    2 P(J)=A(K,J)
      CALL VAGG17(P,QG)
      DO 3 J=1,17
    3 AX(K,J)=QG(J)
    4 CONTINUE
      DO 7 K=1,17
      DO 5 I=1,79
    5 P(I)=AX(I,K)
      CALL VAGG17(P,QG)
      DO 6 I=1,17
    6 AG(I,K)=QG(I)
    7 CONTINUE
      RETURN
      END
```

```
0001      SUBROUTINE VAGG17(A,B)
0002      DIMENSION A(79),B(17)
0003      B( 1)=A( 1)+A( 2)+A( 3)+A( 4)+A(14)+A(15)
0004      B( 2)=A( 9)+A(11)+A(12)+A(35)+A(36)
0005      B( 3)=A( 7)+A( 8)+A(31)+A(68)
0006      B( 4)=A(24)+A(25)+A(26)
0007      B( 5)=A(20)+A(21)+A(22)+A(23)
0008      B( 6)=A(10)+A(16)+A(17)+A(18)+A(19)+A(27)+A(28)+A(29)+A(30)+A(32)+
         1A(33)+A(34)
0009      B( 7)=A( 5)+A(6)+A(37)+A(38)+A(39)+A(40)+A(41)+A(42)
0010      B( 8)=A(13)+A(43)+A(44)+A(45)+A(46)+A(47)+A(48)+A(49)+A(50)+A(51)
         1+A(52)+A(53)+A(54)+A(55)+A(56)+A(57)+A(58)+A(62)+A(63)+A(64)
0011      B( 9)=A(59)+A(60)+A(61)
0012      B(10)=A(65)
0013      B(11)=A(69)
0014      B(12)=A(66)+A(67)
0015      B(13)=A(70)
0016      B(14)=A(71)
0017      B(15)=A(73)+A(74)
0018      B(16)=A(72)+A(75)+A(76)+A(77)
0019      B(17)=A(78)+A(79)
0020      RETURN
0021      END
```

```
FORTRAN IV G LEVEL 1, MOD 3          MAT17          DATE = 69135          05/59/36          PAGE 0001

          SUBROUTINE MAT17(L,A,CHECK)
          DIMENSION L(79),A(17,17)
          MLP=6
       66 FORMAT('1  COL',8X,I2,8(11X,I2),/,' ROW')
       67 FORMAT(1X,I2,5X,9F13.5)
       68 FORMAT(1X,I2,5X,9F13.0)
          LO=1
          IH=9
          WRITE(MLP,66)(L(K),K=LO,IH)
          IF(CHECK-1.0)6,5,6
        5 DO 8 I=1,17
        8 WRITE(MLP,68)I,(A(I,J),J=LO,IH)
          GO TO 9
        6 DO 7 I=1,17
        7 WRITE(MLP,67)I,(A(I,J),J=LO,IH)
        9 LO=10
          IH=17
          WRITE(MLP,66)(L(K),K=LO,IH)
       78 FORMAT(' ROW')
          WRITE(MLP,78)
          IF(CHECK-1.0)12,10,12
       10 DO 11 I=1,17
       11 WRITE(MLP,68)I,(A(I,J),J=LO,IH)
          GO TO 789
       12 DO 13 I=1,17
       13 WRITE(MLP,67)I,(A(I,J),J=LO,IH)
      789 RETURN
          END
```

TABLE E-2

THE MAIN COMPUTER PROGRAM USED IN THE EXPERIMENTS FOR THE "COMMODITY TECHNOLOGY" MODELS

(NOTE: This program refers specifically to the 79x79-order "commodity technology" system. Following the read statements, the dimensions must be changed into 60, 45 or 17, in order to make this program workable at other levels of sectoral aggregation.)

```
FORTRAN IV G LEVEL 1, MOD 3              MAIN              DATE = 69137         14/10/29              PAGE 0001

0001              DIMENSION LLM(79),X(79,79)
0002              DIMENSION B1(79),B2(79),B3(79),B4(79),B5(79),B6(79),B7(79),B8(79),
                 1B9(79),B10(79),B11(79),B12(79),B13(79),B14(79),        B17(79),B18
                 2(79),B19(79),B20(79),B21(79),B7A(79),                  Z(79),ZC(79)
0003              DIMENSION B22(79),B23(79),B24(79),B25(79),B29(79)
0004              DIMENSION P(79),Q(79),SB(2),F(79)
0005              DIMENSION A(79,79),U(79,79),              B26(79)
0006              COMMON A,MLP
0007              MLP=6
0008              IT1=4
            C
            C
0009              N=79
            C
            C
0010              DO 1 I=1,N
0011            1 LLM(I)=I
            C        PUT TAPE IN CORE
0012              DO 2 I=1,N
0013            2 READ(IT1)(A(I,J),J=1,N)
0014              READ(IT1)(Z(I),I=1,N)
0015          301 FORMAT(1X,I2,F13.0)
0016              READ(IT1)(B6(I),I=1,N)
0017              READ(IT1)(B1(I),I=1,N)
0018              READ(IT1)(B2(I),I=1,N)
0019              READ(IT1)(B3(I),I=1,N)
0020              READ(IT1)(B4(I),I=1,N)
0021              READ(IT1)(B5(I),I=1,N)
0022              READ(IT1)(B25(I),I=1,N)
0023              READ(IT1)(B21(I),I=1,N)
0024              READ(IT1)(B22(I),I=1,N)
0025              READ(IT1)(B23(I),I=1,N)
0026              READ(IT1)(B7(I),I=1,N)
0027              READ(IT1)(B7A(I),I=1,N)
0028              READ(IT1)(B8(I),I=1,N)
0029              B8(30)=64.2
0030              B8(45)=2030.2
0031              READ(IT1)(B9(I),I=1,N)
0032              READ(IT1)(B10(I),I=1,N)
0033              B10(8)=10680.0
0034              DO 19 I=1,N
0035           19 B26(I)=B10(I)-B8(I)
0036           20 FORMAT(1X,I2,10X,F14.5,10X,F14.5,10X,F14.5)
0037           21 FORMAT ('1OUTPUT LEVELS 1961',5X,I2,'X1 VECTORS',//)
0038           22 FORMAT('0',17X,'PREDICTED',16X,'ACTUAL',15X,'PRED - ACT',/)
0039              DO 28 I=1,N
0040           28 READ(IT1)(A(I,J),J=1,N)
0041              A(3,33)=934.0
```

```
                A(3,34)=0.0
                A(2,64)=7780.0
                A(26,69)=1203098.0
                A( 1,77)=4552.0
                A( 1,78)=1737.0
                A(25,76)=788.0
                A(26,76)=118766.0
                A( 1,76)=10865.0
                DO 281 J=1,67
                A(73,J)=A(73,J)+A(67,J)
            281 A(67,J)=0.0
                DO 282 J=68,72
                A(73,J)=A(73,J)+A(67,J)
            282 A(67,J)=0.0
                DO 283 J=74,79
                A(73,J)=A(73,J)+A(67,J)
            283 A(67,J)=0.0
             29 FORMAT('1A+M       ** NPA **')
                WRITE(MLP,29)
                CALL MAT79(LLM,A,1.0)
                READ(IT1)(Z(I),I=1,79)
C                         BETTER VALUES FOR **NPA** CONTROL TOTALS
                Z( 1)=23842363.0
                Z( 2)=20735344.0
                Z(12)=16875000.0
                Z(14)=63151574.0
                Z(59)=21846052.0
                Z(65)=32801002.0
                Z(68)=20194431.0
                Z(69)=94350108.0
                Z(70)=25695465.0
                Z(71)=61934295.0
                Z(73)=16962314.0
                Z(77)=22677963.0
            290 FORMAT('1CONTROL TOTALS       ** NPA **',//)
                WRITE(MLP,290)
                DO 291 I=1,N
            291 WRITE(MLP,301)I,Z(I)
                REWIND IT1
                DO 30 J=1,N
                DO 30 I=1,N
             30 A(I,J)=A(I,J)/Z(J)
             31 FORMAT('1COEFFICIENT MATRIX       ** NPA **')
                WRITE(MLP,31)
                CALL MAT79(LLM,A,0.0)
                DO 32 I=1,N
                DO 32 J=1,N
             32 A(I,J)=-A(I,J)
```

```
      DO 33 I=1,N
   33 A(I,I)=A(I,I)+1.0
   34 FORMAT('1I-(A+M)      ** NPA **')
      WRITE(MLP,34)
      CALL MAT79(LLM,A,0.0)
      CALL MINV(A,N,DLL,P,Q)
   35 FORMAT('1A INVERSE      ** NPA **')
      WRITE(MLP,35)
   36 FORMAT('0',10X,'DETERMINANT',2X,F14.5)
      WRITE (MLP,36)DLL
      CALL MAT79(LLM,A,0.0)
      DO 37 I=1,N
      B17(I)=0.0
   37 B18(I)=0.0
      DO 38 I=1,N
      DO 38 J=1,N
   38 B17(I)=A(I,J)*B9(J)+B17(I)
      DO 39 I=1,N
   39 B18(I)=B17(I)-B8(I)
  230 DO 23 I=1,N
   23 B29(I)=B17(I)-B10(I)
      WRITE(MLP,21)N
      WRITE(MLP,24)
   24 FORMAT('0X-CAP D SUB T',/)
      WRITE(MLP,22)
      WRITE(MLP,20) (LLM(I),B17(I),B10(I),B29(I),I=1,N)
      CALL ERROF(B17,B10,N)
      DO 25 I=1,N
   25 B29(I)=B18(I)-B26(I)
      WRITE(MLP,21)N
      WRITE(MLP,26)
   26 FORMAT('0X-CAP D SQ SUB T',/)
      WRITE(MLP,22)
      WRITE(MLP,20) (LLM(I),B18(I),B26(I),B29(I),I=1,N)
      CALL ERROF(B18,B26,N)
  999 CALL EXIT
      END
```

```
FORTRAN IV G LEVEL 1, MOD 3           ERROF              DATE = 69137          14/10/29              PAGE 0001

0001                 SUBROUTINE ERROF(ZC,Z,N)
0002                 DIMENSION E(79),TH(79),ZC(79),X(2,79),Z(79),A(79,79)
                    1,Z1(17),ZC1(17),Z2(17),ZC2(17)
0003                 COMMON A,MLP
             C  **      STEP 1A
0004                 KKK=N
0005              17 DO 1 I=1,N
0006               1 E(I)=ABS(ZC(I)-Z(I))/Z(I)
             C  **      STEP 1B
0007                 E1=0.
0008                 DO 2 I=1,N
0009           2     E1=E(I)+E1
0010                 E1=E1/FLOAT(N)
             C  **      STEP 2
             C  **      COMPUTE MEAN SQUARE PREDICTION ERROR (UNADJUSTED
0011                 S2=0.
0012                 DO 3 I=1,N
0013               3 S2=(ZC(I)-Z(I))**2+S2
0014                 S2=S2/FLOAT(N)
             C  **      PREDICTION STANDARD DEVIATION
0015                 S=SQRT(S2)
             C  **      STANDARD ERROR OF PREDICTION
0016                 SS=SQRT(S2/FLOAT(N))
             C  **      KURTOSIS  PEAKEDNESSS  OF PREDICTION ERRORS
0017                 FK=0.0
0018                 DO 4 I=1,N
0019           4     FK=(ZC(I)-Z(I))**4+FK
0020                 FK=FK/(FLOAT(N)*S2**2)
             C  **      WEIGHTED MEAN SQUARE PREDICTION ERROR
0021                 T1=0.
0022                 T2=0.
0023                 DO 5 I=1,N
0024                 T1=((ZC(I)-Z(I))**2)*Z(I)+T1
0025           5     T2=Z(I)+T2
0026                 SC2=T1/T2
             C  **      PREDICTION STANDARD DEVIATION  WEIGHTED
0027                 SC2=ABS(SC2)
0028                 SC=SQRT(SC2)
             C  **      STANDARD ERROR OF PREDICTION   WEIGHTED
0029                 SS2=SQRT(SC2/FLOAT(N))
0030                 SK=0.
0031                 DO 6 I=1,N
0032           6     SK=(ZC(I)-Z(I))**4+SK
0033                 SK=SK/(FLOAT(N)*SC2**2)
             C  **      LN PREDICTION ERRORS-UNWEIGHTED-
             C  **      LN PREDICTION ERRORS
0034                 DO 7 I=1,N
0035                 IF(ZC(I))72,71,72
```

```
0036          71 TH(I)=0.0
0037             GO TO 7
0038          72 TH(I)=ALOG(ABS(ZC(I)/Z(I)))
0039           7 CONTINUE
0040             THA=0.
0041             DO 8 I=1,N
0042        8    THA=TH(I)+THA
0043             THA=THA/FLOAT(N)
0044             THA=ABS(THA)
           C  **     AVERAGE WEIGHTED LN PREDICTION ERROR
0045             T1=0.
0046             T2=0.
0047             DO 9 I=1,N
0048             T1=TH(I)*Z(I)+T1
0049        9    T2=Z(I)+T2
0050             TS=ABS(T1/T2)
0051             WRITE(MLP,12)
0052       12    FORMAT('1',9X,'GOODNESS OF FIT TESTS',//)
0053             WRITE(MLP,10)E1,S2,S,SS,FK,SC2,SC,SS2
0054       10    FORMAT(' AVERAGE PREDICTION ERROR>',F16.5,/,' MEAN SQUARE PREDICTI
                1ON ERROR UNADJUSTED >',F16.5,/,' PREDICTION STANDARD DEVIATION>',F
                216.5,/,' STANDARD ERROR OF PREDICTION>',F16.5,/,' KURTOSIS PEAKEDN
                3ESS OF PREDICTION ERRORS>',F16.5,/,' WEIGHTED MEAN SQUARE PREDICTI
                4ON ERROR>',F16.5,/,' PREDICTION STANDARD DEVIATION WEIGHTED >',F16
                5.5,/,' STANDARD ERROR OF PREDICTION WEIGHTED >',F16.5)
0055             WRITE(MLP,11)SK,THA,TS
0056       11    FORMAT(' KURTOSIS PEAKEDNESS  OF PREDICTION ERROR>',F16.5,/,' AVER
                1AGE UNWEIGHTED LOGARITHMIC PREDICTION ERROR>',F16.5,/,' AVERAGE WE
                2IGHTED LOGARITHMIC PREDICTION ERROR>',F16.5)
0057             WRITE(MLP,13)
0058       13    FORMAT('1',9X,'E(I)',15X,'LOG PREDICTION ERROR')
0059             DO 14 I=1,N
0060             TH(I)=ABS(TH(I))
0061       14    WRITE(MLP,15)E(I),TH(I)
0062       15    FORMAT(F16.5,4X,F16.5)
0063             IF(N-17)99,99,16
0064          16 CONTINUE
0065             CALL VAGG17(ZC,ZC2)
0066             CALL VAGG17(Z,Z2)
0067             DO 19 I=1,17
0068             Z1(I)=Z(I)
0069             ZC1(I)=ZC(I)
0070             Z(I)=Z2(I)
0071          19 ZC(I)=ZC2(I)
0072             N=17
0073          20 FORMAT(1X,I2,10X,F14.5,10X,F14.5,10X,F14.5)
0074          21 FORMAT('0   AGGREGATED')
0075             WRITE(MLP,21)
```

```
   22 FORMAT('0',17X,'PREDICTED',16X,'ACTUAL',15X,'PRED - ACT',/)
      WRITE(MLP,22)
      DO 18 I=1,17
      E(I)=ZC(I)-Z(I)
   18 WRITE(MLP,20)I,ZC(I),Z(I),E(I)
      GO TO 17
   99 N=KKK
      DO 23 I=1,17
      Z(I)=Z1(I)
   23 ZC(I)=ZC1(I)
      RETURN
      END
```

```
      SUBROUTINE MAT79(L,A,CHECK)
      DIMENSION L(79),A(79,79)
      MLP=6
   66 FORMAT('1  COL.',7X,I2,8(11X,I2),/,' ROW')
   67 FORMAT(1X,I2,5X,9F13.5)
   68 FORMAT(1X,I2,5X,9F13.0)
      LO=1
      IH=9
      DO 77 KK=1,8
      WRITE(MLP,66)(L(K),K=LO,IH)
      IF(CHECK-1.0)6,5,6
    5 DO 8 I=1,79
    8 WRITE(MLP,68)I,(A(I,J),J=LO,IH)
      GO TO 9
    6 DO 7 I=1,79
    7 WRITE(MLP,67)I,(A(I,J),J=LO,IH)
    9 LO=LO+9
   77 IH=IH+9
      LO=73
      IH=79
      WRITE(MLP,66)(L(K),K=LO,IH)
   78 FORMAT(' ROW')
      WRITE(MLP,78)
      IF(CHECK-1.0)786,785,786
  785 DO 788 I=1,79
  788 WRITE(MLP,68)I,(A(I,J),J=LO,IH)
      GO TO 789
  786 DO 787 I=1,79
  787 WRITE(MLP,67)I,(A(I,J),J=LO,IH)
  789 RETURN
      END
```

```
0001    SUBROUTINE VAGG17(A,B)
0002    DIMENSION A(79),B(17)
0003    B( 1)=A( 1)+A( 2)+A( 3)+A( 4)+A(14)+A(15)
0004    B( 2)=A( 9)+A(11)+A(12)+A(35)+A(36)
0005    B( 3)=A( 7)+A( 8)+A(31)+A(68)
0006    B( 4)=A(24)+A(25)+A(26)
0007    B( 5)=A(20)+A(21)+A(22)+A(23)
0008    B( 6)=A(10)+A(16)+A(17)+A(18)+A(19)+A(27)+A(28)+A(29)+A(30)+A(32)+
       1A(33)+A(34)
0009    B( 7)=A( 5)+A(6)+A(37)+A(38)+A(39)+A(40)+A(41)+A(42)
0010    B( 8)=A(13)+A(43)+A(44)+A(45)+A(46)+A(47)+A(48)+A(49)+A(50)+A(51)
       1+A(52)+A(53)+A(54)+A(55)+A(56)+A(57)+A(58)+A(62)+A(63)+A(64)
0011    B( 9)=A(59)+A(60)+A(61)
0012    B(10)=A(65)
0013    B(11)=A(69)
0014    B(12)=A(66)+A(67)
0015    B(13)=A(70)
0016    B(14)=A(71)
0017    B(15)=A(73)+A(74)
0018    B(16)=A(72)+A(75)+A(76)+A(77)
0019    B(17)=A(78)+A(79)
0020    RETURN
0021    END
```

TABLE E-3

THE MATRIX INVERSION SUBROUTINE USED IN THE EXPERIMENTS

```
      $ENTRY
534         SUBROUTINE MINV(A,N,D,L,M)                                  MINV 330
      C  THE ABOVE CARD SHOULD BE PLACED IN PROPER SEQUENCE
      C        BEFORE COMPILING THIS UNDER  IBM FORTRAN G.
      C                                                                 MINV  10
      C     ..................................................................MINV  20
      C                                                                 MINV  30
      C        SUBROUTINE MINV                                          MINV  40
      C                                                                 MINV  50
      C        PURPOSE                                                  MINV  60
      C           INVERT A MATRIX                                       MINV  70
      C                                                                 MINV  80
      C        USAGE                                                    MINV  90
      C           CALL MINV(A,N,D,L,M)                                  MINV 100
      C                                                                 MINV 110
      C        DESCRIPTION OF PARAMETERS                                MINV 120
      C           A - INPUT MATRIX, DESTROYED IN COMPUTATION AND REPLACED BY  MINV 130
      C               RESULTANT INVERSE.                                MINV 140
      C           N - ORDER OF MATRIX A                                 MINV 150
      C           D - RESULTANT DETERMINANT                             MINV 160
      C           L - WORK VECTOR OF LENGTH N                           MINV 170
      C           M - WORK VECTOR OF LENGTH N                           MINV 180
      C                                                                 MINV 190
      C        REMARKS                                                  MINV 200
      C           MATRIX A MUST BE A GENERAL MATRIX                     MINV 210
      C                                                                 MINV 220
      C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED            MINV 230
      C           NONE                                                  MINV 240
      C                                                                 MINV 250
      C        METHOD                                                   MINV 260
      C           THE STANDARD GAUSS-JORDAN METHOD IS USED. THE DETERMINANT  MINV 270
      C           IS ALSO CALCULATED. A DETERMINANT OF ZERO INDICATES THAT   MINV 280
      C           THE MATRIX IS SINGULAR.                               MINV 290
      C                                                                 MINV 300
      C     ..................................................................MINV 310
      C                                                                 MINV 320
535         DIMENSION A(1),L(1),M(1)                                    MINV 340
      C                                                                 MINV 350
      C        ...............................................................MINV 360
      C                                                                 MINV 370
      C        IF A DOUBLE PRECISION VERSION OF THIS ROUTINE IS DESIRED, THE  MINV 380
      C        C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION  MINV 390
      C        STATEMENT WHICH FOLLOWS.                                 MINV 400
      C                                                                 MINV 410
      C     DOUBLE PRECISION A,D,BIGA,HOLD                              MINV 420
      C                                                                 MINV 430
      C        THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS  MINV 440
      C        APPEARING IN OTHER ROUTINES USED IN CONJUNCTION WITH THIS  MINV 450
      C        ROUTINE.                                                 MINV 460
      C                                                                 MINV 470
      C        THE DOUBLE PRECISION VERSION OF THIS SUBROUTINE MUST ALSO  MINV 480
      C        CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS.  ABS IN STATEMENT  MINV 490
      C        10 MUST BE CHANGED TO DABS.                              MINV 500
      C                                                                 MINV 510
      C        ...............................................................MINV 520
```

```
C                                                                       MINV 530
C        SEARCH FOR LARGEST ELEMENT                                     MINV 540
C                                                                       MINV 550
      D=1.0                                                             MINV 560
      NK=-N                                                             MINV 570
      DO 80 K=1,N                                                       MINV 580
      NK=NK+N                                                           MINV 590
      L(K)=K                                                            MINV 600
      M(K)=K                                                            MINV 610
      KK=NK+K                                                           MINV 620
      BIGA=A(KK)                                                        MINV 630
      DO 20 J=K,N                                                       MINV 640
      IZ=N*(J-1)                                                        MINV 650
      DO 20 I=K,N                                                       MINV 660
      IJ=IZ+I                                                           MINV 670
   10 IF( ABS(BIGA)- ABS(A(IJ))) 15,20,20                               MINV 680
   15 BIGA=A(IJ)                                                        MINV 690
      L(K)=I                                                            MINV 700
      M(K)=J                                                            MINV 710
   20 CONTINUE                                                          MINV 720
C                                                                       MINV 730
C        INTERCHANGE ROWS                                               MINV 740
C                                                                       MINV 750
      J=L(K)                                                            MINV 760
      IF(J-K) 35,35,25                                                  MINV 770
   25 KI=K-N                                                            MINV 780
      DO 30 I=1,N                                                       MINV 790
      KI=KI+N                                                           MINV 800
      HOLD=-A(KI)                                                       MINV 810
      JI=KI-K+J                                                         MINV 820
      A(KI)=A(JI)                                                       MINV 830
   30 A(JI) =HOLD                                                       MINV 840
C                                                                       MINV 850
C        INTERCHANGE COLUMNS                                            MINV 860
C                                                                       MINV 870
   35 I=M(K)                                                            MINV 880
      IF(I-K) 45,45,38                                                  MINV 890
   38 JP=N*(I-1)                                                        MINV 900
      DO 40 J=1,N                                                       MINV 910
      JK=NK+J                                                           MINV 920
      JI=JP+J                                                           MINV 930
      HOLD=-A(JK)                                                       MINV 940
      A(JK)=A(JI)                                                       MINV 950
   40 A(JI) =HOLD                                                       MINV 960
C                                                                       MINV 970
C        DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT IS        MINV 980
C        CONTAINED IN BIGA)                                             MINV 990
C                                                                       MINV1000
   45 IF(BIGA) 48,46,48                                                 MINV1010
   46 D=0.0                                                             MINV1020
      RETURN                                                            MINV1030
   48 DO 55 I=1,N                                                       MINV1040
      IF(I-K) 50,55,50                                                  MINV1050
   50 IK=NK+I                                                           MINV1060
      A(IK)=A(IK)/(-BIGA)                                               MINV1070
```

```
   55 CONTINUE
C
C        REDUCE MATRIX
C
      DO 65 I=1,N
      IK=NK+I
      HOLD=A(IK)
      IJ=I-N
      DO 65 J=1,N
      IJ=IJ+N
      IF(I-K) 60,65,60
   60 IF(J-K) 62,65,62
   62 KJ=IJ-I+K
      A(IJ)=HOLD*A(KJ)+A(IJ)
   65 CONTINUE
C
C        DIVIDE ROW BY PIVOT
C
      KJ=K-N
      DO 75 J=1,N
      KJ=KJ+N
      IF(J-K) 70,75,70
   70 A(KJ)=A(KJ)/BIGA
   75 CONTINUE
C
C        PRODUCT OF PIVOTS
C
      D=D*BIGA
C
C        REPLACE PIVOT BY RECIPROCAL
C
      A(KK)=1.0/BIGA
   80 CONTINUE
C
C        FINAL ROW AND COLUMN INTERCHANGE
C
      K=N
  100 K=(K-1)
      IF(K) 150,150,105
  105 I=L(K)
      IF(I-K) 120,120,108
  108 JQ=N*(K-1)
      JR=N*(I-1)
      DO 110 J=1,N
      JK=JQ+J
      HOLD=A(JK)
      JI=JR+J
      A(JK)=-A(JI)
  110 A(JI) =HOLD
  120 J=M(K)
      IF(J-K) 100,100,125
  125 KI=K-N
      DO 130 I=1,N
      KI=KI+N
      HOLD=A(KI)
      JI=KI-K+J
      A(KI)=-A(JI)
  130 A(JI) =HOLD
      GO TO 100
  150 RETURN
      END
```

APPENDIX F

TABLE F-1

PRODUCT-TO-INDUSTRY FLOWS MATRIX, UNITED STATES ECONOMY, 1958

(thousands of 1958 dollars)

$X^{*} = [x_{ij}^{D} + x_{ij}^{M}]^{*} \quad i, j = 1, \ldots, 79.$

COL. / ROW	1	2	3	4	5	6	7	8	9
1	4152571.	1705209.	0.	217710.	0.	0.	0.	0.	0.
2	6599793.	707936.	12142.	4048.	0.	0.	0.	0.	0.
3	0.	0.	15995.	0.	0.	0.	0.	0.	0.
4	452610.	878403.	17461.	1643.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	64029.	8.	0.	0.	0.
6	0.	0.	0.	0.	21154.	231412.	0.	0.	0.
7	5956.	559.	0.	0.	4663.	1266.	470957.	170.	1626.
8	0.	0.	0.	0.	0.	0.	0.	241517.	0.
9	751.	66568.	0.	1.	0.	0.	978.	0.	11803.
10	0.	78322.	66.	0.	0.	0.	66.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	223999.	377096.	330.	1980.	660.	1210.	2413.	4280.	1980.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	3001886.	2816.	24750.	7532.	0.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	7090.	0.	0.	151.	1822.	1836.	0.	0.
17	5789.	26560.	11625.	11899.	0.	0.	0.	2172.	45.
18	0.	0.	0.	0.	0.	0.	0.	0.	0.
19	8299.	34566.	0.	0.	0.	0.	0.	0.	0.
20	1882.	1583.	0.	0.	6006.	1336.	18801.	5734.	30.
21	0.	100172.	0.	73.	0.	0.	0.	0.	0.
22	0.	0.	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.	0.	0.
24	0.	0.	8195.	1972.	27.	513.	6340.	4601.	13579.
25	13934.	2609.	9422.	6148.	0.	0.	1496.	565.	3019.
26	4545.	7215.	94.	25.	47.	404.	731.	839.	509.
27	34261.	1144523.	174.	320.	13098.	41686.	43843.	48712.	17531.
28	0.	0.	0.	0.	0.	0.	0.	0.	0.
29	28754.	0.	0.	50.	6.	107.	0.	748.	189.
30	0.	0.	1851.	0.	11.	199.	877.	4969.	0.
31	48075.	898409.	18891.	2635.	10354.	8215.	30254.	51906.	45393.
32	21219.	155674.	3576.	1832.	630.	3786.	22741.	31294.	25354.
33	0.	0.	0.	0.	0.	0.	0.	0.	0.
34	857.	3674.	15.	2.	3.	4.	8.	45.	5.
35	3405.	0.	0.	0.	0.	0.	0.	426.	0.
36	1042.	25104.	0.	0.	810.	6131.	6284.	3959.	1064.
37	0.	0.	0.	0.	28170.	51075.	22728.	2926.	22327.
38	964.	810.	0.	0.	1846.	6409.	16574.	7195.	1767.
39	5161.	14700.	0.	0.	0.	0.	0.	0.	0.
40	0.	0.	0.	0.	1175.	468.	1036.	5177.	14.
41	21879.	0.	0.	0.	288.	595.	15699.	5594.	223.
42	29037.	38591.	700.	9014.	868.	1354.	14026.	50271.	842.
43	0.	0.	130.	0.	385.	527.	0.	14818.	2.
44	4905.	195578.	0.	0.	0.	0.	0.	0.	22.
45	0.	0.	0.	0.	24019.	28836.	95757.	39456.	75741.
46	0.	0.	0.	0.	20.	139.	10975.	0.	22767.
47	0.	0.	0.	0.	79.	715.	9765.	49.	111.
48	0.	0.	0.	0.	0.	6.	0.	8959.	0.
49	0.	0.	0.	0.	221.	2361.	4804.	89264.	5259.
50	1393.	2599.	0.	0.	73.	92.	551.	710.	501.
51	0.	0.	0.	0.	0.	0.	0.	0.	0.
52	0.	0.	0.	0.	0.	0.	0.	0.	0.
53	0.	0.	0.	0.	775.	4801.	5685.	33086.	2067.
54	0.	0.	0.	0.	0.	0.	0.	0.	0.
55	825.	1003.	104.	0.	375.	506.	4149.	791.	131.
56	0.	0.	0.	0.	1255.	0.	0.	0.	0.
57	0.	0.	0.	0.	0.	0.	0.	11424.	0.
58	7419.	21191.	0.	0.	233.	170.	270.	1132.	251.
59	23805.	34140.	0.	0.	543.	990.	5663.	8978.	6046.
60	0.	0.	0.	0.	0.	0.	0.	0.	0.
61	0.	3009.	20428.	0.	2447.	0.	11501.	0.	274.
62	0.	0.	0.	0.	197.	424.	141.	582.	332.
63	0.	0.	0.	0.	61.	73.	43.	29.	102.
64	1147.	471.	2294.	506.	12.	7.	3925.	448.	503.
65	516562.	269231.	6946.	10926.	24203.	15025.	20588.	83743.	14200.
66	53146.	72158.	3237.	6520.	1604.	1960.	1992.	2441.	2713.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	51228.	172356.	460.	1168.	16938.	29704.	64975.	73452.	35287.
69	511342.	996411.	18349.	4398.	17830.	25565.	95209.	131506.	60790.
70	188822.	236125.	29625.	5278.	6085.	17002.	27240.	103153.	17421.
71	303165.	1809197.	39071.	16176.	78733.	33222.	60543.	1396900.	41701.
72	0.	0.	0.	0.	517.	1081.	655.	425.	1532.
73	46397.	794027.	101897.	4974.	5802.	7893.	13593.	342828.	14303.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	60141.	50617.	0.	0.	0.	0.	980.	14457.	363.
76	0.	0.	0.	0.	22.	26.	14.	10.	33.
77	141956.	12083.	891.	897.	831.	1075.	2704.	9377.	1477.
78	3726.	3137.	521.	1013.	772.	1000.	2738.	0.	1000.
79	468.	713.	95.	176.	87.	370.	617.	4385.	1491.

COL. ROW	1C	11	12	13	14	15	16	17	18
1	0.	0.	0.	C.	14578904.	0.	104151.	54071.	0.
2	0.	236829.	0.	0.	4712225.	1089184.	1178687.	14918.	8427.
3	0.	0.	C.	0.	276090.	0.	0.	0.	140376.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	C.	0.
6	0.	C.	0.	C.	0.	0.	0.	0.	0.
7	468.	58.	0.	0.	40325.	1341.	15658.	1507.	648.
8	0.	50.	C.	0.	0.	0.	0.	C.	0.
9	9242.	625278.	131148.	C.	3633.	0.	0.	50.	0.
10	33077.	0.	0.	0.	8459.	0.	724.	22.	314.
11	0.	0.	C.	C.	0.	0.	0.	0.	0.
12	330.	7000.	1000.	7570.	233450.	320.	6940.	390.	7970.
13	0.	5291.	C.	83043.	0.	0.	0.	0.	0.
14	57.	16801.	0.	0.	10754649.	35267.	23510.	51925.	0.
15	0.	0.	0.	0.	0.	1133233.	0.	0.	0.
16	184.	C.	C.	C.	2234.	1126.	3692620.	348300.	3885824.
17	0.	3866.	1005.	468.	554.	C.	150214.	209057.	78578.
18	0.	0.	C.	7726.	38198.	0.	0.	C.	2455782.
19	0.	355.	724.	C.	99910.	0.	6103.	96.	167249.
20	233.	3280061.	417781.	1105.	3531.	1331.	1241.	25.	0.
21	C.	0.	0.	4476.	98291.	9290.	0.	0.	0.
22	0.	296438.	0.	C.	0.	0.	0.	0.	0.
23	0.	204643.	16470.	0.	0.	C.	0.	0.	0.
24	2972.	323259.	68095.	6908.	371718.	69657.	16550.	14493.	13924.
25	502.	0.	0.	15903.	881906.	70265.	84915.	15017.	38487.
26	27.	7697.	1227.	9448.	122945.	12686.	8079.	1160.	13473.
27	4802.	366665.	70740.	10484.	187244.	5165.	166333.	7209.	42363.
28	0.	0.	0.	0.	11138.	105406.	920002.	389227.	146126.
29	99.	0.	0.	2804.	159216.	7523.	23277.	1056.	1230.
30	0.	196763.	875311.	2265.	5747.	148.	3671.	275.	337.
31	5357.	986432.	374904.	10184.	283812.	2683.	26039.	4216.	5705.
32	2698.	311100.	65922.	113075.	141365.	9837.	40019.	2997.	17335.
33	0.	0.	C.	C.	0.	223.	1791.	299.	37423.
34	3.	316.	52.	55.	246.	8.	43.	0.	85.
35	0.	85252.	81512.	4246.	605707.	0.	23352.	1840.	26.
36	286.	4085008.	548016.	12678.	2302.	29.	514.	67.	100.
37	10388.	2225798.	273666.	50676.	1462.	0.	3930.	766.	935.
38	1810.	868834.	281417.	222132.	35521.	6819.	2554.	533.	0.
39	0.	0.	0.	C.	1525241.	7731.	0.	0.	0.
40	102.	5192256.	881419.	C.	0.	0.	0.	0.	0.
41	429.	38205.	20905.	28899.	174659.	351.	1202.	271.	0.
42	483.	369305.	52195.	46041.	85269.	8911.	8832.	1281.	17229.
43	168.	2120.	354.	4825.	0.	0.	0.	0.	0.
44	0.	2563.	C.	0.	0.	0.	0.	0.	0.
45	14752.	170611.	21220.	0.	0.	C.	0.	0.	0.
46	2333.	249967.	8256.	0.	0.	0.	0.	0.	0.
47	12.	1007.	181.	50168.	13583.	333.	2178.	476.	177.
48	0.	0.	0.	0.	0.	C.	63521.	1568.	0.
49	806.	272914.	18777.	27590.	1691.	201.	1170.	249.	196.
50	69.	2552.	394.	372016.	895.	106.	258.	63.	21.
51	0.	0.	C.	0.	0.	0.	0.	0.	0.
52	0.	196015.	21760.	C.	0.	0.	0.	0.	0.
53	2813.	423920.	78780.	92358.	7946.	46.	595.	138.	138.
54	0.	205565.	59816.	0.	0.	0.	0.	0.	0.
55	27.	734987.	122952.	53888.	22235.	591.	2707.	468.	0.
56	0.	36749.	21674.	142350.	0.	0.	0.	0.	0.
57	0.	1891.	196.	3127.	C.	0.	0.	0.	0.
58	53.	14601.	4896.	368.	3597.	33.	53.	18.	45.
59	1257.	1304.	217.	0.	0.	0.	0.	0.	0.
60	0.	0.	0.	612962.	0.	C.	0.	0.	0.
61	202.	2500.	0.	0.	0.	0.	0.	0.	0.
62	104.	191247.	15843.	98157.	70.	0.	24.	25.	17.
63	32.	5.	1.	261.	C.	0.	0.	0.	0.
64	117.	84057.	48370.	6973.	30340.	6555.	16640.	19002.	255206.
65	15965.	1807429.	298023.	49812.	2561305.	71486.	255645.	40126.	118630.
66	1095.	138419.	18086.	20706.	159024.	2290.	17259.	6070.	42927.
67	0.	0.	C.	0.	0.	0.	0.	0.	0.
68	24172.	150032.	25028.	17652.	355521.	5660.	117635.	16361.	45111.
69	12647.	4962172.	1378711.	135568.	2323214.	77210.	322595.	94064.	504230.
70	3530.	435017.	49233.	30146.	338310.	12012.	63432.	16554.	94543.
71	6072.	209759.	34992.	20592.	290964.	7545.	49226.	16974.	162146.
72	491.	0.	C.	3930.	37763.	2489.	17522.	2784.	36593.
73	4132.	2554327.	60137.	50711.	1602890.	266327.	99347.	17910.	134543.
74	0.	0.	0.	0.	5000.	0.	2000.	C.	0.
75	0.	263865.	22408.	C.	281898.	1838.	6244.	1275.	3393.
76	12.	0.	C.	126.	1022.	50.	349.	61.	755.
77	493.	58065.	9686.	4584.	63951.	5892.	11369.	2175.	16439.
78	465.	0.	0.	3982.	30249.	12604.	8324.	2875.	25804.
79	179.	13252.	2211.	642.	29533.	345.	2452.	503.	1262.

COL. ROW	19	20	21	22	23	24	25	26	27
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	13487.
3	872.	786746.	0.	0.	0.	0.	0.	0.	14787.
4	0.	7921.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	47565.
6	0.	0.	0.	0.	0.	0.	0.	0.	56239.
7	0.	1625.	0.	2129.	0.	73451.	1053.	0.	60695.
8	0.	0.	0.	0.	0.	0.	0.	0.	23657.
9	0.	0.	0.	0.	0.	35426.	0.	0.	19174.
10	0.	76.	0.	0.	0.	15708.	0.	0.	206537.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	340.	15420.	40.	1870.	450.	41580.	13000.	44310.	6080.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	584.	0.	0.	27923.	0.	109295.	0.	0.	118472.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	871384.	0.	0.	187583.	2869.	57485.	0.	0.	939.
17	155381.	0.	0.	37544.	22656.	9316.	0.	17017.	605.
18	1432.	10659.	0.	1120.	1529.	6897.	3246.	0.	5229.
19	142180.	0.	0.	1739.	226.	0.	0.	0.	28565.
20	0.	2393082.	136975.	382853.	81901.	654890.	0.	512.	20383.
21	0.	0.	16421.	432.	0.	5371.	0.	0.	3058.
22	0.	0.	0.	49088.	4714.	0.	0.	0.	0.
23	0.	0.	0.	171.	29644.	0.	0.	0.	0.
24	11969.	57440.	817.	11732.	4069.	1982968.	1547428.	2105566.	61869.
25	21085.	35383.	0.	61029.	27791.	228677.	113063.	43491.	70866.
26	2818.	31635.	956.	1130.	699.	42402.	9455.	1537312.	30283.
27	252.	58425.	8.	208.	234.	333273.	8556.	180964.	2350914.
28	0.	55579.	0.	1125.	1664.	92104.	10231.	0.	24359.
29	666.	9106.	249.	112.	38.	21047.	5597.	5164.	12612.
30	403.	37713.	403.	67110.	24509.	1612.	0.	0.	10155.
31	2145.	75402.	2752.	6994.	3004.	119563.	24813.	10455.	99061.
32	65592.	51784.	423.	126890.	12097.	131460.	23917.	14139.	64633.
33	0.	0.	0.	3827.	2011.	0.	0.	0.	0.
34	12.	41.	3.	24.	13.	48.	20.	317.	132.
35	0.	9860.	0.	46465.	55652.	459.	0.	0.	17888.
36	0.	29424.	680.	8243.	2255.	11191.	224.	141.	32108.
37	0.	332.	17799.	79189.	137667.	0.	1938.	0.	275524.
38	78.	9699.	0.	28077.	19568.	13102.	0.	15172.	101219.
39	0.	0.	0.	0.	0.	0.	0.	0.	83033.
40	0.	0.	0.	0.	0.	0.	0.	0.	0.
41	613.	11688.	944.	11841.	5561.	13432.	3659.	2695.	2239.
42	2115.	63291.	2778.	190658.	51502.	101254.	8415.	11276.	40497.
43	0.	0.	0.	0.	0.	0.	0.	0.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	0.	0.	0.	0.	0.	0.	0.	0.	0.
46	0.	3327.	0.	0.	0.	0.	0.	0.	5696.
47	413.	1266.	142.	3188.	3888.	7942.	1910.	1216.	2187.
48	0.	12523.	780.	3885.	550.	27365.	13323.	34919.	119546.
49	307.	10012.	134.	705.	498.	5438.	1798.	1153.	1059.
50	155.	2594.	142.	355.	250.	3553.	952.	581.	2235.
51	0.	0.	0.	0.	0.	0.	0.	0.	0.
52	0.	0.	0.	0.	0.	0.	0.	0.	0.
53	0.	275.	0.	183.	138.	4445.	367.	321.	13067.
54	0.	0.	0.	0.	0.	0.	0.	0.	0.
55	93.	9748.	887.	1290.	515.	12419.	1557.	1241.	1343.
56	0.	0.	0.	0.	0.	0.	0.	0.	0.
57	0.	0.	0.	0.	0.	0.	0.	0.	0.
58	36.	784.	42.	72.	33.	108.	48.	186.	249.
59	0.	0.	0.	0.	0.	0.	0.	0.	0.
60	0.	0.	0.	0.	0.	0.	0.	0.	0.
61	0.	5571.	0.	0.	0.	0.	0.	0.	0.
62	0.	0.	0.	0.	0.	0.	0.	0.	1305.
63	0.	0.	0.	0.	0.	0.	0.	50135.	531.
64	28605.	3034.	188.	2495.	751.	4606.	2044.	23953.	2562.
65	21351.	383587.	16158.	73050.	25595.	367948.	120308.	186772.	361429.
66	5408.	21728.	932.	17289.	6062.	30002.	7938.	161006.	53219.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	7963.	54678.	3478.	19174.	7879.	183709.	18258.	55175.	301732.
69	104237.	344366.	21884.	182680.	80213.	394974.	134327.	299628.	333713.
70	10119.	52889.	2722.	17516.	7759.	62300.	23088.	172466.	110693.
71	24085.	54574.	4212.	40033.	16062.	42742.	34993.	462002.	121173.
72	3577.	15426.	1146.	6976.	2456.	9989.	4716.	17351.	7566.
73	14311.	46328.	2321.	55216.	14764.	135183.	17193.	563217.	157793.
74	0.	0.	0.	0.	0.	2000.	0.	0.	25000.
75	2777.	65062.	3554.	5524.	2753.	7699.	3268.	13210.	15508.
76	77.	323.	24.	150.	57.	222.	108.	515.	199.
77	2070.	8551.	481.	3611.	1513.	9790.	3830.	13889.	10669.
78	3100.	3321.	407.	1925.	1464.	10970.	3321.	89076.	13475.
79	644.	5376.	229.	615.	244.	12514.	396.	1974.	6163.

ROW \ COL.	28	29	30	31	32	33	34	35	36
1	0.	4834.	0.	0.	0.	208428.	0.	0.	0.
2	0.	7468.	0.	0.	0.	0.	0.	0.	3935.
3	0.	0.	0.	0.	0.	977.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	2610.	0.	0.	0.	0.	8975.
6	0.	480.	166.	0.	0.	0.	0.	95.	4036.
7	22671.	3851.	552.	9572.	9980.	2200.	0.	2871.	62794.
8	0.	0.	0.	8956743.	0.	0.	0.	0.	0.
9	497.	3666.	856.	48807.	5975.	0.	0.	24747.	439214.
10	452.	44.	0.	1106.	7595.	852.	0.	642.	10328.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	26480.	1720.	710.	25430.	6500.	80.	400.	960.	3490.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	17464.	145245.	56466.	10988.	381.	824285.	3048.	0.	5254.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	3999.	678.	391.	27.	142981.	0.	80048.	0.	12669.
17	0.	0.	0.	0.	433716.	0.	41489.	119.	1631.
18	1549.	2325.	751.	2951.	8088.	0.	7451.	2634.	0.
19	0.	182.	0.	0.	997.	1404.	587.	0.	2084.
20	953.	0.	0.	811.	4157.	0.	25832.	30704.	11768.
21	895.	0.	0.	0.	0.	1876.	979.	11762.	11680.
22	0.	0.	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.	0.	0.
24	193226.	55508.	14030.	56152.	19302.	2705.	25859.	4919.	164165.
25	23475.	200819.	20260.	32509.	49045.	1327.	37681.	166772.	72627.
26	4150.	26992.	4466.	699.	14402.	1729.	10336.	3282.	15345.
27	1139789.	618812.	405411.	465962.	306352.	62631.	1003.	83584.	151309.
28	114586.	753.	226484.	0.	907792.	0.	2049.	0.	83458.
29	35384.	413131.	12363.	38542.	5298.	22772.	1023.	3293.	44964.
30	0.	8052.	2937.	0.	240.	22.	43.	97.	3687.
31	20058.	24156.	9378.	1242226.	15546.	3584.	974.	6798.	63497.
32	11263.	60035.	5541.	5841.	207227.	4412.	174627.	9356.	41664.
33	0.	0.	0.	0.	0.	126984.	646580.	0.	0.
34	6.	32.	17.	58.	50.	3.	259172.	17.	52.
35	1157.	132419.	0.	2321.	43079.	0.	260.	106442.	411.
36	725.	11941.	8437.	26049.	11796.	7427.	7217.	58423.	912285.
37	0.	693.	10483.	3988.	12199.	0.	0.	0.	58119.
38	1474.	897.	546.	1887.	3029.	182.	338.	4776.	14334.
39	5269.	107541.	87499.	121742.	0.	0.	0.	0.	0.
40	0.	0.	0.	0.	0.	0.	0.	0.	0.
41	3237.	21239.	723.	1849.	6080.	578.	1632.	9134.	6781.
42	5753.	52048.	2594.	188254.	70999.	1400.	23045.	8537.	37726.
43	0.	0.	0.	0.	0.	0.	0.	0.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	0.	0.	0.	0.	0.	0.	0.	0.	0.
46	0.	0.	0.	0.	0.	0.	0.	0.	0.
47	5572.	804.	191.	2546.	3195.	614.	284.	3150.	3083.
48	5562.	3504.	0.	0.	0.	0.	0.	0.	1162.
49	2269.	220.	320.	794.	2992.	112.	255.	2837.	1610.
50	1143.	110.	160.	373.	17053.	113.	257.	1420.	1735.
51	0.	0.	0.	0.	0.	0.	0.	0.	0.
52	0.	0.	0.	0.	0.	0.	0.	0.	0.
53	2729.	275.	46.	3072.	3234.	517.	92.	1438.	6554.
54	0.	0.	0.	0.	0.	0.	0.	0.	0.
55	1617.	268.	311.	483.	7291.	822.	551.	1575.	[illegible]
56	0.	0.	0.	0.	0.	0.	0.	0.	0.
57	0.	0.	0.	0.	0.	0.	0.	0.	0.
58	27.	63.	47.	336.	39.	9.	15.	21.	386.
59	0.	0.	0.	0.	0.	0.	0.	0.	0.
60	0.	0.	0.	0.	0.	0.	0.	0.	0.
61	0.	0.	0.	0.	0.	0.	0.	0.	0.
62	791.	5600.	233.	915.	2448.	285.	0.	1069.	[illegible]
63	187.	223.	72.	283.	586.	76.	0.	252.	714.
64	2526.	3525.	1338.	740.	2506.	520.	3617.	951.	4114.
65	129184.	124204.	54770.	837083.	148399.	20527.	37035.	51384.	447645.
66	11429.	13960.	7661.	23636.	20847.	2654.	9156.	5523.	25805.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	40734.	26579.	8287.	253783.	70887.	6975.	10336.	73944.	212587.
69	82506.	174817.	78339.	171088.	234117.	23167.	84341.	77495.	223255.
70	29561.	52224.	14195.	113986.	39425.	5381.	22935.	20868.	74567.
71	24532.	55936.	22482.	138188.	71970.	2782.	28436.	15923.	62461.
72	2817.	3373.	1031.	0.	8843.	1146.	9007.	3795.	10908.
73	63692.	997697.	27890.	351703.	163015.	4038.	87792.	33798.	156987.
74	12000.	0.	0.	7000.	0.	0.	0.	2000.	0.
75	1585.	3921.	2910.	18705.	2471.	592.	1162.	1619.	24858.
76	73.	104.	35.	0.	207.	24.	183.	91.	249.
77	3858.	6383.	1880.	16261.	7065.	931.	3608.	2344.	7946.
78	19591.	14119.	4250.	31642.	5700.	4586.	6989.	5643.	6571.
79	1360.	1492.	434.	6642.	1867.	329.	269.	692.	2952.

COL. / ROW	37	38	39	40	41	42	43	44	45
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	3100.	0.
5	1046135.	19721.	0.	0.	0.	0.	0.	0.	0.
6	6997.	798037.	0.	0.	0.	1683.	0.	0.	0.
7	506997.	12262.	372.	1762.	1348.	1768.	2256.	2352.	2039.
8	0.	0.	0.	0.	0.	0.	0.	0.	0.
9	54644.	4416.	0.	1410.	0.	1086.	0.	400.	372.
10	7517.	1860.	0.	0.	0.	264.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	127880.	3610.	530.	0.	2360.	3580.	700.	1810.	720.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	6793.	633.	0.	0.	0.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	15138.	0.	733.	295.	5446.	201.	147.	251.
17	807.	5187.	0.	310.	0.	5575.	473.	0.	0.
18	14909.	5141.	0.	7160.	3897.	6729.	1527.	1807.	2320.
19	2084.	1904.	0.	0.	0.	0.	0.	0.	0.
20	16856.	10272.	852.	11331.	18627.	45301.	600.	8997.	3537.
21	3251.	0.	997.	7820.	1748.	2246.	0.	234.	0.
22	0.	0.	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.	1518.	0.
24	48251.	27695.	9119.	13719.	16057.	7905.	2458.	1482.	2803.
25	15280.	7026.	28000.	31655.	31526.	33173.	10514.	5442.	2692.
26	28521.	9028.	6298.	3990.	2770.	4288.	3243.	1163.	1305.
27	173491.	80356.	0.	19692.	12013.	63024.	124.	2453.	2490.
28	4728.	112839.	2034.	2112.	14802.	4560.	1425.	381.	1458.
29	35595.	9720.	4920.	2109.	1920.	1050.	980.	1479.	1588.
30	15216.	7339.	35031.	21175.	18393.	9950.	2628.	9754.	4467.
31	147203.	41828.	5647.	42476.	23493.	29036.	8761.	9766.	13546.
32	61621.	14534.	30520.	10051.	20964.	39144.	10744.	82657.	45806.
33	0.	0.	784.	1033.	433.	617.	317.	2673.	415.
34	82.	38.	6.	49.	30.	57.	17.	19.	26.
35	1165.	299.	46.	29997.	3630.	1652.	1169.	283.	91.
36	255546.	43356.	4129.	44444.	27723.	44653.	14354.	13560.	16836.
37	4420238.	24925.	919944.	1827469.	666929.	816013.	223719.	361540.	471302.
38	275116.	2984858.	18057.	546483.	231025.	364565.	70913.	18666.	22045.
39	0.	0.	3249.	0.	0.	0.	0.	0.	0.
40	0.	0.	0.	150251.	920.	0.	107.	0.	23574.
41	114574.	85827.	19385.	116985.	97999.	87567.	41757.	71634.	31021.
42	301578.	91942.	6860.	203368.	61004.	247877.	5554.	10412.	[illegible]
43	0.	0.	0.	866.	5513.	0.	201662.	119241.	77947.
44	0.	0.	0.	0.	0.	0.	0.	92121.	2629.
45	0.	0.	0.	0.	0.	0.	0.	2389.	172530.
46	0.	0.	0.	0.	0.	0.	2717.	0.	0.
47	112620.	65046.	25686.	37695.	27159.	152736.	35578.	38958.	42306.
48	14432.	0.	0.	70.	512.	912.	0.	0.	0.
49	48988.	27365.	14257.	65481.	81.	12523.	46751.	137410.	155342.
50	141891.	32866.	9926.	25573.	11369.	11356.	66253.	35652.	12952.
51	0.	0.	0.	0.	0.	0.	0.	0.	0.
52	0.	0.	0.	13519.	0.	0.	0.	0.	0.
53	81349.	10206.	2156.	78464.	6481.	22283.	19120.	13142.	32226.
54	0.	0.	0.	366.	831.	0.	0.	4409.	0.
55	3972.	39639.	836.	7534.	5770.	4290.	964.	1683.	1515.
56	0.	0.	0.	0.	0.	0.	0.	0.	0.
57	0.	0.	0.	0.	928.	2714.	0.	0.	0.
58	102.	87.	18.	252.	48.	93.	36787.	18119.	4750.
59	0.	0.	0.	0.	0.	0.	2909.	15545.	1167.
60	0.	0.	0.	0.	978.	0.	1939.	0.	0.
61	0.	0.	0.	0.	0.	0.	0.	0.	0.
62	5091.	2041.	407.	64408.	1316.	3163.	1705.	3094.	922.
63	1428.	492.	118.	661.	360.	621.	141.	173.	214.
64	11702.	6785.	1513.	5142.	1739.	3665.	2380.	1458.	1667.
65	582760.	193427.	53020.	142605.	56656.	85947.	27245.	39893.	44106.
66	68095.	27851.	2102.	32240.	9496.	17653.	5618.	6180.	5876.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	455498.	216849.	13908.	49912.	33064.	47649.	9456.	13643.	18825.
69	683635.	312691.	81047.	290185.	107635.	219241.	59077.	99614.	109696.
70	143222.	60492.	12987.	63981.	29236.	43129.	13020.	15911.	21634.
71	74807.	39334.	8631.	49578.	30995.	35516.	9955.	13135.	16931.
72	21584.	7435.	1801.	5585.	5437.	9400.	2129.	2620.	3242.
73	168949.	78176.	19505.	91387.	33894.	94401.	25334.	65619.	42547.
74	17000.	5000.	0.	2000.	0.	0.	12000.	0.	0.
75	6693.	5948.	835.	16540.	2997.	6396.	633.	2533.	2629.
76	479.	173.	38.	245.	125.	221.	57.	65.	86.
77	19875.	9206.	2131.	7986.	3651.	6110.	2082.	2466.	3566.
78	15393.	4714.	1464.	8096.	3793.	6364.	2168.	4714.	2343.
79	13567.	2903.	197.	1780.	997.	1528.	320.	773.	451.

COL. / ROW	46	47	48	49	50	51	52	53	54
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	586.	0.	0.	0.	0.	0.	3228.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	0.	0.	34.	748.	209.	0.	746.	1677.	1185.
8	0.	0.	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	0.	9104.	41.	0.	0.	0.	0.
10	0.	0.	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	350.	7730.	2420.	4360.	6520.	1620.	2700.	6760.	2260.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	0.	0.	0.	0.	0.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	1626.	0.	3143.	1737.	0.	138.	253.	1488.	9101.
17	0.	0.	0.	0.	0.	7.	0.	1573.	256.
18	784.	3943.	2545.	3499.	2072.	1900.	1591.	4636.	2545.
19	0.	0.	0.	0.	0.	0.	0.	0.	0.
20	833.	5065.	12542.	5160.	0.	1199.	5848.	6482.	5695.
21	15.	15.	355.	214.	0.	753.	11026.	145.	20490.
22	0.	0.	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.	0.	0.
24	699.	0.	4390.	9916.	135.	14097.	8685.	40223.	5396.
25	394.	1030.	704.	6183.	0.	3876.	18705.	19972.	34541.
26	611.	227.	433.	2025.	241.	265.	358.	6022.	2304.
27	0.	3523.	3924.	5136.	0.	763.	14370.	18067.	18292.
28	531.	1203.	1827.	1010.	0.	2585.	2243.	35219.	9329.
29	746.	1433.	1693.	1074.	912.	463.	531.	569.	636.
30	2730.	170.	813.	2511.	0.	2174.	11176.	14740.	20760.
31	3624.	18310.	14228.	13877.	16029.	3350.	6615.	16009.	4595.
32	20433.	1294.	20363.	14565.	1829.	22983.	25926.	36155.	117362.
33	106.	263.	3064.	922.	507.	227.	311.	557.	873.
34	11.	34.	28.	39.	14.	37.	19.	79.	4528.
35	329.	490.	86.	73.	47.	567.	3711.	4531.	4467.
36	4102.	24780.	9959.	28472.	24646.	4966.	13361.	39520.	28010.
37	116253.	244773.	219648.	374565.	117921.	41820.	153739.	294707.	273751.
38	12641.	70402.	102471.	89417.	113544.	41726.	113792.	323239.	151285.
39	0.	0.	0.	0.	0.	0.	0.	0.	0.
40	8595.	3589.	19301.	21578.	0.	0.	6082.	11575.	9146.
41	13578.	27465.	25085.	38665.	3185.	24530.	78040.	74737.	173296.
42	10961.	13723.	29689.	48552.	18464.	11798.	49394.	34137.	111955.
43	8217.	0.	236.	10911.	0.	0.	2505.	8053.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	1379.	0.	0.	0.	0.	0.	0.	0.	0.
46	40598.	3306.	52.	0.	0.	0.	0.	0.	0.
47	17423.	206854.	29051.	54550.	23746.	20666.	5644.	52441.	26844.
48	0.	7549.	125350.	143.	0.	0.	0.	0.	53.
49	51288.	98440.	106666.	257311.	3191.	13289.	11970.	42055.	31342.
50	15215.	5216.	8322.	13065.	107032.	4679.	1972.	8120.	3489.
51	101.	0.	1808.	0.	0.	193021.	0.	0.	0.
52	0.	0.	37.	0.	0.	0.	105724.	622.	66794.
53	50617.	84053.	84793.	133139.	7481.	45614.	193147.	341123.	129983.
54	0.	0.	0.	0.	0.	0.	130.	0.	36568.
55	1234.	760.	617.	2523.	1211.	8968.	14507.	40890.	25557.
56	0.	0.	0.	0.	0.	836.	0.	102.	0.
57	0.	0.	0.	612.	0.	92431.	144.	103425.	0.
58	2471.	33.	51.	670.	27.	6.	33.	33.	0.
59	0.	0.	0.	0.	0.	0.	0.	0.	0.
60	0.	0.	0.	0.	0.	0.	0.	0.	0.
61	0.	0.	0.	0.	0.	0.	0.	0.	0.
62	349.	1355.	3029.	16041.	1089.	1004.	27557.	5795.	106339.
63	75.	465.	244.	464.	198.	175.	147.	428.	235.
64	498.	1955.	1302.	1055.	502.	3130.	2545.	2234.	2591.
65	12893.	28423.	28188.	49395.	15474.	18410.	33606.	60745.	56848.
66	3718.	25665.	24257.	33679.	10799.	9419.	7783.	20557.	14879.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	4263.	22212.	13955.	22698.	12482.	7398.	11192.	35019.	21700.
69	45694.	102384.	96724.	169857.	48818.	107350.	121957.	154527.	151374.
70	8351.	27987.	17754.	23093.	12593.	11476.	18063.	22711.	11720.
71	10039.	63637.	24694.	26450.	23559.	17256.	28988.	39700.	18860.
72	1146.	5599.	3701.	4783.	3013.	2653.	2227.	6495.	3570.
73	16246.	37737.	30267.	43811.	16291.	57686.	21771.	56799.	307005.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	729.	2498.	3692.	2715.	1955.	586.	2485.	2506.	663.
76	33.	139.	96.	126.	68.	73.	59.	173.	95.
77	958.	3424.	2488.	3576.	1659.	1936.	2096.	5124.	3605.
78	1333.	2857.	2393.	3786.	1929.	3431.	2393.	12635.	7492.
79	89.	421.	467.	599.	284.	215.	361.	882.	671.

COL. ROW	55	56	57	58	59	60	61	62	63
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	4326.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	1948.	0.	0.	0.	0.	0.
6	3337.	0.	0.	1591.	0.	0.	0.	1494.	0.
7	0.	1775.	431.	0.	14820.	2254.	2607.	0.	2468.
8	0.	0.	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	0.	260.	126.	0.	265.	110.	0.
10	0.	0.	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	130.	8350.	2000.	100.	71070.	22710.	1380.	550.	520.
13	0.	988.	0.	0.	0.	199066.	0.	0.	0.
14	0.	0.	0.	0.	0.	0.	0.	11538.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	1478.	0.	506.	50548.	1854.	2274.	27594.	819.
17	0.	0.	0.	0.	88237.	6204.	2668.	6296.	468.
18	2234.	4496.	3531.	1433.	10363.	11296.	4032.	3299.	1333.
19	0.	0.	0.	0.	149994.	0.	1553.	0.	0.
20	1637.	10842.	691.	0.	11060.	20236.	88103.	1200.	107.
21	0.	0.	0.	0.	0.	0.	0.	1031.	0.
22	0.	141290.	14250.	0.	0.	0.	24433.	0.	0.
23	0.	0.	0.	0.	4491.	16118.	14338.	0.	0.
24	6368.	26742.	12970.	1978.	83270.	5032.	6271.	15463.	49903.
25	40494.	31252.	14699.	10829.	24478.	9494.	1261.	27681.	10507.
26	736.	7718.	1415.	326.	12247.	10101.	1821.	1951.	392.
27	12536.	6151.	30334.	32914.	40929.	15659.	8437.	9164.	23799.
28	42638.	41778.	20266.	8577.	26382.	8577.	27092.	8522.	1087.
29	387.	2020.	155.	109.	18544.	5140.	2260.	617.	109.
30	10938.	2639.	1664.	9.	88912.	10390.	29734.	1836.	310.
31	4127.	7460.	4223.	2128.	47698.	30302.	16350.	4639.	4129.
32	35106.	59192.	17566.	70353.	623438.	79102.	33179.	39415.	11439.
33	379.	1063.	134.	106.	6510.	0.	1627.	2587.	106.
34	22.	81.	40.	15.	89.	137.	34.	2146.	753.
35	66260.	27366.	84312.	992.	233960.	1607.	19033.	11231.	16252.
36	13311.	18958.	17400.	14436.	61715.	44694.	30757.	12305.	4403.
37	143909.	55710.	50064.	44636.	2003727.	405318.	432340.	67426.	9222.
38	107581.	117256.	117772.	154094.	260184.	340898.	68349.	137870.	42294.
39	0.	0.	0.	0.	0.	0.	0.	3448.	0.
40	0.	2863.	205.	0.	7329.	805.	155123.	2580.	0.
41	59012.	111922.	61687.	36361.	699570.	244460.	17731.	62639.	9345.
42	34946.	71568.	34740.	5821.	817252.	119936.	73281.	39472.	15569.
43	0.	0.	0.	0.	61578.	1833.	99020.	0.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	0.	0.	0.	0.	1045.	0.	17802.	0.	0.
46	0.	0.	0.	0.	0.	0.	12942.	0.	0.
47	12154.	25632.	15171.	20572.	255877.	237469.	18295.	40397.	6705.
48	0.	0.	0.	0.	0.	0.	0.	0.	0.
49	234.	1911.	1643.	25567.	106065.	122733.	55714.	13712.	0.
50	3669.	5584.	2700.	3293.	134457.	125189.	13259.	17870.	0.
51	0.	0.	0.	0.	0.	795.	0.	20407.	0.
52	0.	3789.	0.	0.	27408.	989.	8930.	409.	0.
53	35935.	56015.	10907.	12275.	28657.	35740.	122757.	71737.	13554.
54	0.	0.	0.	0.	0.	0.	18975.	0.	0.
55	92027.	62643.	11125.	26336.	91053.	15481.	11986.	10652.	4718.
56	0.	332234.	2759.	0.	112892.	308442.	5906.	15593.	0.
57	0.	1033444.	161686.	10658.	17037.	74316.	350.	95133.	0.
58	44592.	21.	0.	57133.	326350.	45363.	4513.	231.	15.
59	0.	0.	0.	6831.	6795292.	40892.	28396.	0.	0.
60	0.	0.	0.	0.	0.	2414187.	0.	4386.	0.
61	0.	0.	0.	0.	2870.	937.	251304.	0.	0.
62	807.	11353.	2546.	1056.	97552.	160511.	5143.	201627.	1014.
63	206.	415.	336.	131.	992.	25504.	383.	3642.	85643.
64	6766.	2740.	2356.	824.	19682.	7863.	8069.	13679.	664.
65	30612.	69056.	27939.	19783.	426175.	112038.	65147.	33953.	22175.
66	6077.	23150.	7507.	5015.	47379.	56478.	10307.	15611.	4776.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	12133.	16159.	19663.	9470.	108232.	69043.	23619.	13469.	5725.
69	143783.	210542.	144879.	52100.	687523.	228987.	177747.	132827.	55275.
70	9969.	20211.	12681.	6206.	86788.	37763.	18989.	17743.	2439.
71	21590.	36424.	53323.	11230.	63248.	72105.	16920.	35091.	20244.
72	3111.	6288.	5076.	1965.	15000.	0.	5764.	4693.	1932.
73	30319.	137846.	26851.	32900.	574065.	43760.	32320.	52002.	85251.
74	0.	2000.	2000.	0.	12000.	12000.	0.	0.	2000.
75	710.	1340.	48.	710.	8176.	1697.	3422.	383.	994.
76	73.	172.	118.	46.	343.	0.	128.	126.	52.
77	2337.	6056.	2740.	1467.	23048.	11510.	3925.	3415.	1599.
78	3473.	14372.	8710.	3555.	38810.	11946.	2857.	3995.	2303.
79	424.	589.	483.	346.	4118.	1995.	830.	491.	154.

COL. / ROW	64	65	66	67	68	69	70	71	72
1	0.	1635.	0.	0.	0.	0.	0.	0.	0.
2	7334.	7909.	0.	0.	0.	0.	0.	19829.	0.
3	2876.	1420.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	152547.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	750.	26600.	0.	0.	543276.	0.	5996.	0.	0.
8	0.	0.	0.	0.	1087755.	0.	0.	0.	0.
9	376.	1345.	0.	0.	0.	0.	0.	4045.	0.
10	39.	872.	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	15570.	1248686.	293000.	8000.	550817.	775000.	119671.	5898905.	35000.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	6518.	100098.	0.	0.	0.	2156.	0.	0.	13154.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	58157.	6679.	692.	0.	0.	0.	0.	0.	125167.
17	33783.	15596.	3495.	1747.	0.	16385.	21243.	0.	23196.
18	4129.	4330.	0.	0.	0.	0.	324.	0.	96669.
19	1813.	15137.	5140.	2570.	0.	21921.	31247.	0.	125717.
20	92117.	9103.	157.	0.	0.	4179.	0.	0.	4952.
21	1814.	17365.	0.	0.	0.	89000.	0.	0.	0.
22	0.	0.	0.	0.	0.	0.	0.	0.	11125.
23	0.	0.	0.	0.	0.	0.	0.	0.	3379.
24	110577.	33558.	0.	8292.	8937.	432049.	101658.	0.	145544.
25	201304.	8480.	0.	1983.	0.	346532.	24309.	0.	18835.
26	12138.	75325.	110240.	9750.	5154.	174093.	393787.	775.	6597.
27	37811.	35470.	273.	0.	111.	0.	0.	17122.	98695.
28	121033.	108.	0.	0.	0.	0.	0.	0.	0.
29	4884.	11542.	934.	42.	422.	0.	11146.	0.	177093.
30	39352.	39762.	0.	0.	0.	0.	0.	0.	102.
31	16785.	1518410.	16232.	2006.	241449.	687548.	93522.	245619.	143728.
32	118755.	257355.	6248.	857.	7851.	156356.	53168.	33400.	68902.
33	33025.	3291.	0.	0.	0.	0.	0.	0.	0.
34	25649.	393.	250.	40.	154.	3641.	991.	214.	4774.
35	26504.	5411.	0.	0.	0.	101469.	0.	0.	3136.
36	10396.	3869.	0.	0.	23043.	41904.	0.	0.	51746.
37	144029.	37750.	0.	0.	2711.	0.	0.	0.	0.
38	247574.	48661.	22837.	0.	0.	0.	0.	0.	6353.
39	0.	0.	0.	0.	0.	3561.	0.	0.	0.
40	0.	552.	0.	0.	0.	0.	0.	0.	0.
41	41937.	15281.	0.	0.	0.	19637.	0.	0.	0.
42	81457.	40997.	2903.	0.	158056.	9309.	0.	0.	27860.
43	0.	81227.	0.	0.	0.	0.	0.	0.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	0.	0.	0.	0.	0.	0.	0.	0.	0.
46	0.	13929.	0.	0.	0.	0.	0.	0.	0.
47	2154.	23468.	130.	0.	2265.	0.	0.	0.	0.
48	20.	0.	0.	0.	0.	0.	0.	0.	622.
49	3355.	14387.	0.	0.	0.	0.	0.	0.	0.
50	11096.	6959.	0.	0.	0.	12915.	0.	0.	0.
51	0.	2493.	0.	0.	7981.	7385.	4671.	0.	0.
52	1500.	2164.	0.	0.	0.	4983.	0.	0.	39992.
53	28807.	30843.	413.	0.	9733.	0.	0.	0.	4629.
54	0.	0.	0.	0.	0.	0.	0.	0.	117939.
55	14126.	7060.	0.	0.	2509.	3541.	0.	0.	7430.
56	0.	17639.	115618.	35081.	3615.	2629.	0.	0.	[illegible]
57	8680.	25005.	5277.	0.	0.	0.	0.	0.	232777.
58	114.	73102.	4424.	0.	363.	21872.	1919.	[illegible]	2831.
59	0.	88015.	0.	0.	0.	163331.	0.	0.	0.
60	978.	163394.	0.	0.	0.	0.	0.	0.	0.
61	0.	303056.	4761.	0.	1166.	0.	9523.	0.	5012.
62	807.	25865.	49.	0.	1.	0.	0.	0.	70015.
63	0.	37.	3.	0.	0.	0.	0.	0.	105818.
64	291894.	47573.	8969.	8969.	6143.	44081.	22422.	0.	258191.
65	81248.	2106453.	16471.	2549.	367727.	349311.	235010.	26267.	38154.
66	29180.	271383.	82967.	61308.	46666.	1012309.	411725.	171060.	64984.
67	0.	0.	0.	4150.	0.	0.	0.	0.	0.
68	26405.	147176.	56960.	6274.	3375988.	1916086.	125139.	230519.	234954.
69	317217.	1004057.	48996.	18232.	239967.	1581646.	255777.	167474.	519578.
70	43808.	701727.	52553.	12570.	110601.	1579803.	5389299.	1295634.	107694.
71	79062.	1023106.	152494.	77960.	55713.	5048166.	2058932.	1267307.	555307.
72	9556.	0.	0.	0.	0.	224811.	0.	0.	353921.
73	123522.	497752.	162932.	62217.	225298.	4755078.	1024976.	707833.	315101.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	7804.	820503.	15168.	1685.	24763.	825861.	83045.	44717.	115467.
76	217.	25003.	2918.	300620.	0.	95507.	5000.	0.	0.
77	5641.	31460.	8626.	1496.	16500.	95408.	140440.	12602.	11721.
78	9599.	43030.	38455.	917.	49875.	415154.	326527.	244132.	7022.
79	1285.	55579.	3787.	141.	6082.	50336.	7343.	2323.	18638.

"INDUSTRY" OUTPUT

COL. ROW	73	74	75	76	77	78	79
1	C.C	0.0	0.0	10821.00000	4503.00000	1738.00000	0.C
2	0.0	0.0	0.0	0.0	5334.00000	621609.00000	0.0
3	C.C	0.0	0.0	0.0	C.C	0.0	0.0
4	0.C	0.0	0.0	3100.00000	0.0	0.0	560.00000
5	0.0	0.0	0.0	0.C	0.0	1751.00000	0.0
6	C.C	0.0	0.0	0.0	0.0	0.0	0.C
7	29960.00000	0.0	10498.CCCCC	0.0	213.00000	46803.00000	79273.00000
8	0.0	0.0	0.0	0.0	C.0	0.0	19938.00000
9	C.0	C.0	0.0	0.0	0.0	399.00000	0.0
10	0.0	0.0	C.0	C.C	0.0	0.0	0.0
11	C.C	0.0	0.0	0.0	C.C	0.0	C.C
12	22000.CCCCC	C.C	108217.000CC	126000.00000	680426.00000	14172.00000	192000.00000
13	0.0	0.0	0.0	C.0	0.0	0.0	10.00000
14	C.C	0.0	0.0	0.0	169628.00000	259384.0C000	535.C0000
15	0.0	0.0	0.0	0.C	174.00000	0.0	0.0
16	0.0	0.0	0.0	0.0	2433.00CCC	0.0	0.0
17	13978.CCCCC	1747.C0000	16237.00000	3228.00000	22713.00000	0.0	2158.00000
18	4333.00000	433.C0000	1141.0000C	0.C	38203.CC000	0.0	1671.00000
19	21647.CCCCC	257C.00000	18340.0000C	5439.00000	34345.C0C0C	2011.00000	0.0
20	C.C	0.0	0.C	0.0	3222.00000	0.0	41.00000
21	0.0	0.0	0.0	C.C	0.0	0.0	0.0
22	C.C	C.C	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.C	0.0	0.0	0.0	0.0
24	65862.00000	6777.CC000	3274.CC00C	3385.CCCCC	84715.00000	27109.00000	1895.00000
25	8103.C0000	1620.00000	942.00000	810.00000	20264.00CCC	6483.CC000	0.C
26	65785.CCCC0	468.CCCC0	10177.CC00C	19831.C00C0	322403.00000	37758.00000	12023.00000
27	13749.00000	0.0	64.0000C	0.0	4456.0C000	0.0	23788.C0000
28	C.C	0.0	0.0	0.0	0.0	0.0	37.C0000
29	27878.00000	56C7.00000	9346.CC0CC	934.C00CC	588190.00000	0.0	2863.00000
30	0.C	0.0	61143.00000	0.0	0.C	0.0	113.00000
31	108437.C0CCC	323.C000C	27226.00000	3711.C0000	70070.00000	6158.00000	40397.00000
32	65529.00000	17152.CCC00	271868.C000C	1715.00C0C	64275.CC000	1072.00000	5461.00000
33	C.0	C.0	0.0	0.0	0.0	0.0	0.0
34	489.00000	0.0	50.0C0CC	9772.CCCCC	2577.C0000	746.C0000	0.0
35	C.C	0.0	97037.00000	0.0	6253.00000	0.0	0.C
36	220.C0000	0.0	39266.CCCCC	0.0	76.00000	11799.00000	610.00000
37	0.0	0.0	0.0	C.C	0.0	0.0	2258.C0000
38	15737.CCCC0	C.C	0.0	0.0	0.0	0.0	0.0
39	0.0	0.C	0.0	C.0	C.C	0.0	0.0
40	C.C	0.0	0.0	0.0	0.C	0.0	0.0
41	C.C	C.C	0.0	0.0	20308.00000	0.0	0.0
42	765.C0CCC	0.0	113535.C0C00	C.0	392.00000	2960.00000	21492.00000
43	102286.C0000	0.0	0.0	C.C	0.0	0.0	0.0
44	135758.CC000	0.0	0.0	0.0	0.0	0.0	0.0
45	17772.C0000	0.0	C.C	C.0	0.0	0.0	0.0
46	0.0	0.0	0.0	0.0	0.C	0.0	0.0
47	184CC.CCCCC	93C.00000	960.00000	0.0	0.0	0.0	0.0
48	0.0	0.0	0.C	0.C	0.0	0.0	0.0
49	C.C	0.0	0.0	0.0	0.C	0.0	C.C
50	398.C0CCC	C.C	104898.00CCC	0.0	136.00000	739.00000	1033.00000
51	8C31C.CCCCC	0.0	0.0	0.0	2988.00000	0.0	0.0
52	85982.00000	1505.C0C00	C.C	0.0	0.0	0.0	0.0
53	92.00000	0.0	6115.C0CCC	C.C	0.0	0.0	0.0
54	C.C	0.C	0.0	0.0	0.0	0.0	0.0
55	109.00000	C.C	33183.CCCCC	C.C	37.00000	203.00000	282.00000
56	C.C	0.0	0.0	0.0	10997.0CCCC	0.0	0.0
57	C.C	C.C	0.0	0.0	0.0	0.0	0.0
58	4656.CC000	C.0	117816.C00CC	0.0	11657.00000	513.00000	950.00000
59	5C16.00000	0.0	1131299.00000	0.0	173C.CC0CC	935C.0C000	12304.00000
60	0.C	0.0	0.0	C.C	C.C	0.0	0.0
61	18223.CCCCC	5714.00000	9523.0000C	952.0000C	19607.00000	0.0	0.0
62	0.C	0.C	16C58.CCCCC	0.0	259727.00000	0.0	0.0
63	191905.00000	0.0	0.0	24691.CCCCC	72207.CCCCC	0.0	0.0
64	73304.C0CCC	4933.CC00C	1794.00000	86755.00000	32056.00000	0.0	340.C0CCC
65	112727.00000	20C7.C0CC0	74267.C0CCC	22536.C0000	119946.CC0CC	731389.00000	69593.00000
66	272161.CCCC0	2000.00000	55215.00000	31078.0000C	190203.0C00C	11325.0C0CC	24997.0000C
67	3452.CCCCC	0.C	0.0	0.0	0.0	0.0	0.0
68	246014.00000	0.0	151306.000CC	42334.0CCCC	417753.C000C	72785.00000	376252.00000
69	421790.00000	11628.CCCCC	671384.CCCCC	7C482.C0000	42011C.00000	61960.00000	42120.00000
70	293862.00000	5000.00000	207191.00000	129652.00000	264568.C0000	5C59.CC000	45000.C0000
71	895146.C0000	11637.CC00C	310C68.C0000	285400.00000	1557075.00000	40671.00000	71120.00000
72	0.0	0.0	0.0	C.0	110CCC.C0000	0.0	7412.00000
73	548232.C0000	20000.00000	153924.00000	213074.00000	572535.00000	55608.00000	68175.C0000
74	C.C	0.0	C.C	0.0	40000.00000	1304.00000	0.0
75	117289.00000	0.0	133166.C0CCC	C.C	48719.00000	35532.CC000	4839.00000
76	C.C	5CCC.C0000	0.0	1328499.00000	83420.00000	0.0	0.0
77	3578.00000	516.C0000	7623.C0CCC	5423.C0000	296092.CC000	0.0	220.00000
78	656859.00000	0.0	3564.00000	3139.0000C	15893.CCCCC	6324.00CC0	4090.C0000
79	7288.CCC00	0.0	6C9.00000	1215.00000	12293.00000	875.00000	485.00000

CONTROL TOTALS

COL.	
1	26129136.
2	23043440.
3	922383.
4	1018326.
5	783877.
6	1C16674.
7	2748IC9.
8	9668889.
9	14C4415.
10	468553.
11	52416000.
12	16874392.
13	4148456.
14	62527632.
15	5517774.
16	1C481097.
17	2064326.
18	142C116C.
19	1856001.
20	7641575.
21	4C6882.
22	3177285.
23	1351155.
24	933C889.
25	3558331.
26	12449234.
27	10529355.
28	37658CC.
29	6219967.
30	1815079.
31	1687376C.
32	6540772.
33	1646367.
34	3C5C575.
35	2122914.
36	7326252.
37	19153824.
38	8888412.
39	2060487.
40	7363CC9.
41	3315068.
42	55C8974.
43	1954167.
44	2313021.
45	2865335.
46	9C6151.
47	3080627.
48	2253941.
49	3753364.
50	1473195.
51	2131773.
52	1942805.
53	4678583.
54	3421897.
55	2151483.
56	5635C74.
57	2353748.
58	1376766.
59	22559264.
60	1195021C.
61	3601448.
62	3C65954.
63	1461860.
64	5C31555.
65	32568672.
66	9292068.
67	1548537.
68	1717736C.
69	92203232.
70	2640IC24.
71	55274304.
72	12169295.
73	1644781C.
74	53427C.
75	7891796.
76	5619749.
77	2270324B.
78	41C5C41.
79	47839CC.

TABLE F-2

"INDUSTRY TECHNOLOGY" MATRIX, UNITED STATES ECONOMY, 1958

$$(A^D + M)^* = [a^D_{ij} + m_{ij}]^* \quad i, j = 1, \ldots, 79.$$

COL. / ROW	1	2	3	4	5	6	7	8	9
1	0.15892	0.07400	0.0	0.21379	0.0	0.0	0.0	0.0	0.0
2	0.25258	0.03072	0.01316	0.00398	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.01734	0.0	0.0	0.0	0.0	0.0	0.0
4	0.01885	0.03812	0.01893	0.00161	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.08168	0.00001	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.02699	0.22762	0.0	0.0	0.0
7	0.00023	0.00002	0.0	0.0	0.00595	0.00125	0.17137	0.00002	0.00116
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.02458	0.0
9	0.00003	0.00289	0.0	0.00000	0.0	0.0	0.00036	0.0	0.00840
10	0.0	0.00123	0.00007	0.0	0.0	0.0	0.00002	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00896	0.01636	0.00036	0.00194	0.00084	0.00119	0.00088	0.00044	0.00141
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.11489	0.00012	0.02683	0.00740	0.0	0.0	0.0	0.0	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.00031	0.0	0.0	0.00019	0.00179	0.00067	0.0	0.0
17	0.00022	0.00115	0.01260	0.01168	0.0	0.0	0.0	0.00022	0.00003
18	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
19	0.00032	0.00152	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.00007	0.00007	0.0	0.0	0.00766	0.00131	0.00684	0.00059	0.00002
21	0.0	0.00435	0.0	0.00007	0.0	0.0	0.0	0.0	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.0	0.0	0.00987	0.00194	0.00003	0.00050	0.00231	0.00048	0.00967
25	0.00053	0.00011	0.01021	0.00604	0.0	0.0	0.00054	0.00006	0.00215
26	0.00019	0.00031	0.00010	0.00002	0.00006	0.00040	0.00027	0.00009	0.00036
27	0.00131	0.04567	0.00019	0.00031	0.01671	0.04100	0.01595	0.00504	0.01248
28	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
29	0.00110	0.0	0.0	0.00005	0.00001	0.00011	0.0	0.00008	0.00013
30	0.0	0.0	0.00201	0.0	0.00001	0.00020	0.00032	0.00051	0.0
31	0.00184	0.03899	0.02048	0.00259	0.01321	0.00808	0.01101	0.00537	0.03232
32	0.00081	0.00676	0.00930	0.00180	0.00080	0.00372	0.00828	0.00322	0.02696
33	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
34	0.00003	0.00016	0.00002	0.00000	0.00000	0.00000	0.00000	0.00002	0.00000
35	0.00013	0.0	0.0	0.0	0.0	0.0	0.0	0.00004	0.0
36	0.00004	0.00109	0.0	0.0	0.00103	0.00603	0.00229	0.00041	0.00076
37	0.0	0.0	0.0	0.0	0.03594	0.05024	0.00827	0.00030	0.01590
38	0.00004	0.00004	0.0	0.0	0.00235	0.00630	0.00603	0.00074	0.00126
39	0.00020	0.00064	0.0	0.0	0.0	0.0	0.0	0.0	0.0
40	0.0	0.0	0.0	0.0	0.00150	0.00046	0.00038	0.00064	0.00001
41	0.00084	0.0	0.0	0.0	0.00037	0.00059	0.00571	0.00058	0.00017
42	0.00111	0.00167	0.00076	0.00885	0.00111	0.00133	0.00510	0.00520	0.00067
43	0.0	0.0	0.00014	0.0	0.00049	0.00052	0.0	0.00153	0.00000
44	0.00019	0.00849	0.0	0.0	0.0	0.0	0.0	0.0	0.00002
45	0.0	0.0	0.0	0.0	0.03064	0.02841	0.03630	0.00408	0.05353
46	0.0	0.0	0.0	0.0	0.00003	0.00014	0.00399	0.0	0.01621
47	0.0	0.0	0.0	0.0	0.00010	0.00070	0.00355	0.00001	0.00008
48	0.0	0.0	0.0	0.0	0.0	0.00001	0.0	0.00053	0.0
49	0.0	0.0	0.0	0.0	0.00028	0.00232	0.00175	0.00830	0.00371
50	0.00007	0.00012	0.0	0.0	0.00009	0.00009	0.00020	0.00007	0.00036
51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
53	0.0	0.0	0.0	0.0	0.00099	0.00472	0.00207	0.00342	0.00211
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
55	0.00003	0.00004	0.00011	0.0	0.00048	0.00050	0.00151	0.00007	0.00009
56	0.0	0.0	0.0	0.0	0.00160	0.0	0.0	0.0	0.0
57	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00118	0.0
58	0.00128	0.00091	0.0	0.0	0.00030	0.00017	0.00010	0.00012	0.00020
59	0.00091	0.00148	0.0	0.0	0.00069	0.00097	0.00206	0.00093	0.00430
60	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.0	0.00013	0.02215	0.0	0.00312	0.0	0.00419	0.0	0.00020
62	0.0	0.0	0.0	0.0	0.00025	0.00042	0.00005	0.00006	0.00023
63	0.0	0.0	0.0	0.0	0.00008	0.00007	0.00002	0.00000	0.00007
64	0.00004	0.00002	0.00249	0.00050	0.00002	0.00001	0.00143	0.00005	0.00042
65	0.01977	0.01168	0.00753	0.01073	0.03088	0.01478	0.00749	0.00866	0.01011
66	0.00203	0.00313	0.00351	0.00640	0.00205	0.00193	0.00072	0.00125	0.00153
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.00349	0.00743	0.00050	0.00115	0.02161	0.02922	0.02364	0.00911	0.02797
69	0.03438	0.04324	0.01989	0.00432	0.02275	0.02515	0.03465	0.01059	0.04329
70	0.00723	0.01242	0.03212	0.00511	0.00776	0.01672	0.00991	0.01119	0.01255
71	0.01160	0.07851	0.04236	0.01588	0.10044	0.03760	0.02203	0.14447	0.02904
72	0.0	0.0	0.0	0.0	0.00117	0.00106	0.00024	0.00004	0.00110
73	0.00178	0.03446	0.11047	0.00488	0.00740	0.00776	0.00495	0.03546	0.01019
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.00230	0.00220	0.0	0.0	0.0	0.0	0.00036	0.00159	0.00026
76	0.0	0.0	0.0	0.0	0.00003	0.00003	0.00001	0.00000	0.00002
77	0.00543	0.00052	0.00097	0.00088	0.00106	0.00106	0.00098	0.00097	0.00105
78	0.00014	0.00014	0.00100	0.00099	0.00098	0.00098	0.00100	0.0	0.00071
79	0.00002	0.00003	0.00010	0.00017	0.00011	0.00036	0.00022	0.00045	0.00126

COL. / ROW	10	11	12	13	14	15	16	17	18
1	0.0	0.0	0.0	0.0	0.23956	0.0	0.00994	0.02619	0.0
2	0.0	0.00452	0.0	0.0	0.07536	0.18405	0.11246	0.00723	0.00059
3	0.0	0.0	0.0	0.0	0.00442	0.0	0.0	0.0	0.00988
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.00100	0.00000	0.0	0.0	0.00064	0.00023	0.00149	0.00073	0.00005
8	0.0	0.00000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.01972	0.01193	0.00777	0.0	0.00006	0.0	0.0	0.00002	0.0
10	0.07059	0.0	0.0	0.0	0.00014	0.0	0.00007	0.00001	0.00002
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00070	0.00013	0.00006	0.00182	0.00373	0.00005	0.00066	0.00010	0.00056
13	0.0	0.00010	0.0	0.02002	0.0	0.0	0.0	0.0	0.0
14	0.00012	0.00032	0.0	0.0	0.17200	0.00596	0.00224	0.02515	0.0
15	0.0	0.0	0.0	0.0	0.0	0.19150	0.0	0.0	0.0
16	0.00039	0.0	0.0	0.0	0.00004	0.00019	0.35231	0.16872	0.27363
17	0.0	0.00007	0.00006	0.00011	0.00001	0.0	0.01433	0.10127	0.00553
18	0.0	0.0	0.0	0.00066	0.00061	0.0	0.0	0.0	0.17293
19	0.0	0.00001	0.00004	0.0	0.00160	0.0	0.00058	0.00005	0.01178
20	0.00050	0.06258	0.02476	0.00027	0.00006	0.00022	0.00012	0.00001	0.0
21	0.0	0.0	0.0	0.00108	0.00157	0.00157	0.0	0.0	0.0
22	0.0	0.00566	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.00390	0.00098	0.0	0.0	0.0	0.0	0.0	0.0
24	0.00634	0.00617	0.00404	0.00167	0.00594	0.01177	0.00158	0.00702	0.00098
25	0.00107	0.0	0.0	0.00383	0.01410	0.01187	0.00810	0.00727	0.00623
26	0.00006	0.00015	0.00007	0.00228	0.00197	0.00214	0.00077	0.00056	0.00025
27	0.01025	0.00700	0.00419	0.00253	0.00299	0.00087	0.01587	0.00349	0.00298
28	0.0	0.0	0.0	0.0	0.00018	0.01781	0.08778	0.18855	0.01029
29	0.00021	0.0	0.0	0.00068	0.00255	0.00127	0.00222	0.00051	0.00009
30	0.0	0.00375	0.05211	0.00055	0.00009	0.00003	0.00035	0.00013	0.00002
31	0.01150	0.01382	0.02222	0.00245	0.00454	0.00045	0.00248	0.00204	0.00049
32	0.00576	0.00594	0.00391	0.02726	0.00226	0.00166	0.00382	0.00145	0.00122
33	0.0	0.0	0.0	0.0	0.0	0.00004	0.00017	0.00014	0.00264
34	0.00001	0.00001	0.00000	0.00001	0.00000	0.00000	0.00000	0.00000	0.00001
35	0.0	0.00163	0.00483	0.00102	0.00969	0.0	0.00223	0.00082	0.00000
36	0.00061	0.07793	0.03248	0.00306	0.00004	0.00000	0.00005	0.00003	0.00001
37	0.02217	0.04246	0.01622	0.01222	0.00002	0.0	0.00037	0.00037	0.00007
38	0.00386	0.01658	0.01668	0.05355	0.00057	0.00115	0.00024	0.00026	0.0
39	0.0	0.0	0.0	0.0	0.02439	0.00131	0.0	0.0	0.0
40	0.00022	0.09906	0.05223	0.0	0.0	0.0	0.0	0.0	0.0
41	0.00092	0.00168	0.00124	0.00697	0.00279	0.00006	0.00011	0.00013	0.0
42	0.00103	0.01658	0.00309	0.01110	0.00135	0.00151	0.00084	0.00062	0.00121
43	0.00036	0.00004	0.00002	0.00116	0.0	0.0	0.0	0.0	0.0
44	0.0	0.00005	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.03148	0.00325	0.00126	0.0	0.0	0.0	0.0	0.0	0.0
46	0.00638	0.00477	0.00049	0.0	0.0	0.0	0.0	0.0	0.0
47	0.00003	0.00002	0.00001	0.01209	0.00022	0.00006	0.00021	0.00023	0.00001
48	0.0	0.0	0.0	0.0	0.0	0.0	0.00606	0.00076	0.0
49	0.00172	0.00521	0.00111	0.00665	0.00003	0.00003	0.00011	0.00012	0.00001
50	0.00021	0.00025	0.00002	0.08968	0.00001	0.00002	0.00002	0.00003	[illegible]
51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.00374	0.00129	0.0	0.0	0.0	0.0	0.0	0.0
53	0.00600	0.00809	0.00467	0.02226	0.00013	0.00001	0.00026	0.00007	0.00001
54	0.0	0.03392	0.00354	0.0	0.0	0.0	0.0	0.0	0.0
55	0.00006	0.01517	0.00729	0.01299	0.00036	0.00010	0.00026	0.00023	0.0
56	0.0	0.00070	0.00128	0.03431	0.0	0.0	0.0	0.0	0.0
57	0.0	0.00004	0.00001	0.00075	0.0	0.0	0.0	0.0	0.0
58	0.00012	0.07228	0.00029	0.00005	0.00006	0.00001	0.00001	0.00001	[illegible]
59	0.00268	0.00002	0.00001	0.0	0.0	0.0	0.0	0.0	0.0
60	0.0	0.0	0.0	0.14776	0.0	0.0	0.0	0.0	0.0
61	0.00043	0.00035	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.00022	0.00365	0.00094	0.02125	0.00000	0.0	0.00000	0.00001	0.00003
63	0.00007	0.00000	0.00000	0.00006	0.0	0.0	0.0	0.0	0.0
64	0.00025	0.00160	0.00287	0.00168	0.00049	0.00111	0.00159	0.00920	0.01797
65	0.03407	0.03443	0.01766	0.01201	0.04096	0.01208	0.02439	0.02380	0.00835
66	0.00233	0.00207	0.00107	0.00499	0.00254	0.00039	0.00165	0.00294	0.00303
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.05159	0.00286	0.00148	0.00426	0.00569	0.00096	0.01122	0.00793	0.00318
69	0.02699	0.09467	0.08170	0.03268	0.03715	0.01305	0.03078	0.04557	0.03551
70	0.00753	0.03830	0.00292	0.00727	0.00541	0.00203	0.00605	0.00802	0.07464
71	0.01296	0.00400	0.00207	0.00496	0.00465	0.00127	0.00470	0.00817	0.01142
72	0.00105	0.0	0.0	0.00055	0.00060	0.00042	0.00167	0.01135	0.00259
73	0.00882	0.04873	0.00356	0.01222	0.02563	0.04500	0.00948	0.00868	0.00247
74	0.0	0.0	0.0	0.0	0.00008	0.0	0.00019	0.0	0.0
75	0.0	0.00503	0.00133	0.0	0.00451	0.00031	0.00060	0.00062	0.00024
76	0.00003	0.0	0.0	0.00003	0.00002	0.00001	0.00003	0.00003	0.00005
77	0.00106	0.00111	0.00057	0.00110	0.00102	0.00100	0.00108	0.00105	0.00116
78	0.00093	0.0	0.0	0.00056	0.00048	0.00213	0.00079	0.00139	0.00182
79	0.00038	0.00025	0.00013	0.00015	0.00047	0.00006	0.00023	0.00024	0.00009

CCL. ROW	19	20	21	22	23	24	25	26	27
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00128
3	0.00047	0.10296	0.0	0.0	0.0	0.0	0.0	0.0	0.00140
4	0.0	0.00104	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00452
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00534
7	0.0	0.00021	0.0	0.00067	0.0	0.00787	0.00030	0.0	0.00576
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00225
9	0.0	0.0	0.0	0.0	0.0	0.00380	0.0	0.0	0.00182
10	0.0	0.00001	0.0	0.0	0.0	0.00168	0.0	0.0	0.01962
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00018	0.00202	0.00010	0.00059	0.00033	0.00446	0.00365	0.00356	0.00057
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.00031	0.0	0.0	0.00879	0.0	0.01171	0.0	0.0	0.01125
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.36173	0.0	0.0	0.05904	0.00212	0.00616	0.0	0.0	0.00009
17	0.08372	0.0	0.0	0.01182	0.01677	0.00100	0.0	0.00137	0.00006
18	0.00077	0.00139	0.0	0.00035	0.00113	0.00074	0.00051	0.0	0.00050
19	0.07661	0.0	0.0	0.00055	0.00017	0.0	0.0	0.0	0.00271
20	0.0	0.31317	0.33500	0.12050	0.06062	0.07019	0.0	0.00004	0.00279
21	0.0	0.0	0.04016	0.00014	0.0	0.00058	0.0	0.0	0.00029
22	0.0	0.0	0.0	0.01545	0.00349	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.00005	0.02194	0.0	0.0	0.0	0.0
24	0.00645	0.00752	0.00200	0.00369	0.00301	0.21252	0.43487	0.16913	0.00873
25	0.01185	0.00470	0.0	0.01921	0.02057	0.02451	0.03177	0.00349	0.00673
26	0.00152	0.00414	0.00234	0.00036	0.00052	0.00454	0.00266	0.12831	0.00287
27	0.00014	0.00765	0.00002	0.00007	0.00017	0.03572	0.00240	0.01454	0.22327
28	0.0	0.00727	0.0	0.00035	0.00123	0.00987	0.00288	0.0	0.00279
29	0.00036	0.00119	0.00061	0.00004	0.00003	0.00226	0.00157	0.00041	0.00120
30	0.00022	0.00494	0.00099	0.02112	0.01814	0.00017	0.0	0.0	0.00096
31	0.00116	0.00987	0.00673	0.00220	0.00222	0.01281	0.00697	0.00084	0.00941
32	0.03534	0.00678	0.00103	0.03994	0.00895	0.01409	0.00672	0.00114	0.00614
33	0.0	0.0	0.0	0.00120	0.00149	0.0	0.0	0.0	0.0
34	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00003	0.00001
35	0.0	0.00129	0.0	0.01462	0.04119	0.00005	0.0	0.0	0.00170
36	0.0	0.00385	0.00166	0.00259	0.00167	0.00120	0.00006	0.00001	0.00305
37	0.0	0.00004	0.04353	0.02492	0.10189	0.0	0.00054	0.0	0.02522
38	0.00004	0.00127	0.0	0.00884	0.01448	0.00140	0.0	0.00122	0.00961
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00797
40	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
41	0.00033	0.00153	0.00231	0.00373	0.00412	0.00144	0.00103	0.00022	0.00028
42	0.00168	0.00828	0.00679	0.06001	0.03812	0.01085	0.00236	0.00001	0.00399
43	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
46	0.0	0.00044	0.0	0.0	0.0	0.0	0.0	0.0	0.00054
47	0.00022	0.00017	0.00035	0.00100	0.00288	0.00085	0.00054	0.00010	0.00021
48	0.0	0.00164	0.00191	0.00122	0.00041	0.00293	0.00374	0.00280	0.01097
49	0.00017	0.00131	0.00033	0.00022	0.00037	0.00058	0.00051	0.00009	0.00019
50	0.00008	0.00034	0.00035	0.00011	0.00019	0.00038	0.00027	0.00005	0.00031
51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
53	0.0	0.00004	0.0	0.00006	0.00010	0.00048	0.00010	0.00003	0.00124
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
55	0.00005	0.00128	0.00217	0.00041	0.00038	0.00133	0.00044	0.00010	0.00013
56	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
57	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
58	0.00002	0.00010	0.00010	0.00007	0.00002	0.00001	0.00001	0.00001	0.00002
59	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
60	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.0	0.00073	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00019
63	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00403	0.00005
64	0.01541	0.00040	0.00046	0.00079	0.00056	0.00049	0.00057	0.00192	0.00024
65	0.01150	0.05025	0.03952	0.02299	0.01894	0.03943	0.03381	0.01500	0.03433
66	0.00291	0.00284	0.00233	0.00544	0.00449	0.00322	0.00223	0.01293	0.00505
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.00424	0.00716	0.00851	0.00603	0.00583	0.02022	0.00513	0.00443	0.02866
69	0.05616	0.04506	0.05352	0.05750	0.05937	0.04233	0.03775	0.02407	0.03169
70	0.00545	0.00692	0.00666	0.00551	0.00574	0.00668	0.00649	0.00984	0.01049
71	0.01298	0.00714	0.01030	0.01260	0.01189	0.00458	0.00993	0.03711	0.01151
72	0.00191	0.00202	0.00280	0.00220	0.00182	0.00107	0.00133	0.00147	0.00072
73	0.00771	0.00613	0.00568	0.01738	0.01093	0.01449	0.00483	0.04524	0.01499
74	0.0	0.0	0.0	0.0	0.0	0.00021	0.0	0.0	0.00237
75	0.00150	0.00851	0.00869	0.00174	0.00204	0.00083	0.00092	0.00106	0.00151
76	0.00004	0.00004	0.00006	0.00005	0.00004	0.00002	0.00003	0.00004	0.00002
77	0.00112	0.00112	0.00118	0.00114	0.00112	0.00105	0.00108	0.00113	0.00101
78	0.00167	0.00043	0.00100	0.00061	0.00108	0.00118	0.00093	0.00716	0.00128
79	0.00035	0.00070	0.00056	0.00019	0.00018	0.00134	0.00011	0.00016	0.00059

COL. ROW	28	29	30	31	32	33	34	35	36
1	0.0	0.00078	0.0	0.0	0.0	0.12660	0.0	0.0	0.0
2	0.0	0.00040	0.0	0.0	0.0	0.0	0.0	0.0	0.00054
3	0.0	0.0	0.0	0.0	0.0	0.00059	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.00015	0.0	0.0	0.0	0.0	0.00123
6	0.0	0.00008	0.00009	0.0	0.0	0.0	0.0	0.00004	0.00055
7	0.00602	0.00062	0.00030	0.00057	0.00153	0.00134	0.0	0.00135	0.00857
8	0.0	0.0	0.0	0.53081	0.0	0.0	0.0	0.0	0.0
9	0.00013	0.00059	0.00047	0.00289	0.00091	0.0	0.0	0.01166	0.05995
10	0.00013	0.00001	0.0	0.00007	0.00116	0.00052	0.0	0.00030	0.00141
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00703	0.00028	0.00039	0.00151	0.00099	0.00005	0.00013	0.00045	0.00048
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.00464	0.02335	0.03111	0.00065	0.00006	0.50067	0.00100	0.0	0.00072
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.00196	0.00011	0.00022	0.00000	0.02186	0.0	0.02624	0.0	0.00172
17	0.0	0.0	0.0	0.0	0.06631	0.0	0.01360	0.00006	0.00022
18	0.00052	0.00037	0.00041	0.00017	0.00124	0.0	0.00244	0.00124	0.0
19	0.0	0.00003	0.0	0.0	0.00015	0.00085	0.00019	0.0	0.00028
20	0.00025	0.0	0.0	0.00005	0.00064	0.0	0.00847	0.01446	0.00161
21	0.00024	0.0	0.0	0.0	0.0	0.00114	0.00032	0.00554	0.00151
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.05131	0.00892	0.00773	0.00333	0.00295	0.00164	0.00848	0.00232	0.02241
25	0.00623	0.03229	0.01116	0.00193	0.00750	0.00081	0.01235	0.07856	0.00991
26	0.00111	0.00434	0.00246	0.00004	0.00220	0.00105	0.00335	0.00155	0.00209
27	0.30267	0.09949	0.22336	0.02761	0.04684	0.03804	0.00033	0.03937	0.02065
28	0.03043	0.00012	0.12478	0.0	0.13879	0.0	0.00067	0.0	0.01139
29	0.00940	0.06642	0.00681	0.00228	0.00081	0.01383	0.00034	0.00155	0.00614
30	0.0	0.00129	0.00162	0.0	0.00004	0.00001	0.00001	0.00005	0.00050
31	0.00533	0.00388	0.00517	0.07362	0.00238	0.00218	0.00032	0.00320	0.00867
32	0.00299	0.00965	0.00305	0.00035	0.03168	0.00268	0.05724	0.00394	0.00569
33	0.0	0.0	0.0	0.0	0.0	0.07713	0.21195	0.0	0.0
34	0.00000	0.00001	0.00001	0.00000	0.00001	0.00000	0.08496	0.00001	0.00001
35	0.00031	0.02129	0.0	0.00014	0.00659	0.0	0.00009	0.05014	0.00006
36	0.00019	0.00192	0.00465	0.00154	0.00180	0.00451	0.00237	0.02752	0.12466
37	0.0	0.00011	0.00578	0.00024	0.00187	0.0	0.0	0.0	0.00793
38	0.00039	0.00014	0.00030	0.00011	0.00046	0.00011	0.00011	0.00225	0.00196
39	0.00135	0.01729	0.04821	0.00721	0.0	0.0	0.0	0.0	0.0
40	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
41	0.00086	0.00341	0.00040	0.00011	0.00093	0.00035	0.00053	0.00430	0.00093
42	0.00151	0.50837	0.00143	0.01116	0.01085	0.00085	0.00755	0.00402	0.01197
43	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
46	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
47	0.00148	0.00013	0.00011	0.00015	0.00049	0.00037	0.00009	0.00148	0.00054
48	0.00143	0.00056	0.0	0.0	0.0	0.0	0.0	0.0	0.00016
49	0.00060	0.00004	0.00018	0.00005	0.00046	0.00007	0.00008	0.00134	0.00022
50	0.00083	0.00002	0.00009	0.00002	0.00261	0.00007	0.00008	0.00067	0.00023
51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
53	0.00072	0.00004	0.00003	0.00018	0.00049	0.00031	0.00003	0.00068	0.00080
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
55	0.00043	0.00004	0.00017	0.00003	0.00111	0.00050	0.00018	0.00074	0.00358
56	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
57	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
58	0.00001	0.00001	0.00002	0.00002	0.00001	0.00001	0.00000	0.00001	0.00005
59	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
60	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.00021	0.00090	0.00013	0.00005	0.00037	0.00017	0.0	0.00050	0.00036
63	0.00005	0.00004	0.00004	0.00002	0.00005	0.00005	0.0	0.00012	0.00010
64	0.00057	0.00057	0.00074	0.00004	0.00038	0.00032	0.00119	0.00045	0.00056
65	0.03430	0.01997	0.03017	0.04561	0.02269	0.01247	0.01214	0.02420	0.06102
66	0.00303	0.00224	0.00422	0.00140	0.00319	0.00161	0.00300	0.00307	0.00352
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.01383	0.00427	0.00457	0.01504	0.01084	0.00424	0.00339	0.03766	0.02902
69	0.02191	0.02311	0.04319	0.01014	0.03579	0.01407	0.02765	0.03650	0.03047
70	0.00783	0.00840	0.00782	0.00676	0.00603	0.00327	0.00752	0.00983	0.01018
71	0.00651	0.00899	0.01239	0.00819	0.01100	0.00169	0.00932	0.00750	0.00825
72	0.00075	0.00054	0.00060	0.0	0.00135	0.00070	0.00295	0.00179	0.00148
73	0.01425	0.16040	0.01537	0.02084	0.02492	0.00245	0.02878	0.01592	0.01460
74	0.00317	0.0	0.0	0.00041	0.0	0.0	0.0	0.00094	0.0
75	0.00042	0.00063	0.00160	0.00111	0.00038	0.00036	0.00038	0.00076	0.00339
76	0.00002	0.00002	0.00002	0.0	0.00003	0.00001	0.00006	0.00004	0.00003
77	0.00104	0.00103	0.00104	0.00096	0.00108	0.00057	0.00118	0.00110	0.00108
78	0.00520	0.00227	0.00234	0.00158	0.00087	0.00279	0.00229	0.00266	0.00090
79	0.00036	0.00024	0.00024	0.00039	0.00029	0.00020	0.00009	0.00042	0.00039

COL. ROW	37	38	39	40	41	42	43	44	45
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00134	0.0
5	0.05462	0.00222	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.00037	0.08978	0.0	0.0	0.0	0.00031	0.0	0.0	0.0
7	0.02647	0.00138	0.00018	0.00024	0.00041	0.00032	0.00115	0.00102	0.00071
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.00285	0.00050	0.0	0.00019	0.0	0.00020	0.0	0.00017	0.00013
10	0.00039	0.00021	0.0	0.0	0.0	0.00005	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00668	0.00041	0.00026	0.0	0.00071	0.00065	0.00036	0.00078	0.00025
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.00035	0.00007	0.0	0.0	0.0	0.0	0.0	0.0	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.00170	0.0	0.00010	0.00009	0.00099	0.00010	0.00006	0.00009
17	0.00004	0.00058	0.0	0.00004	0.0	0.00101	0.00024	0.0	0.0
18	0.00078	0.00058	0.0	0.00097	0.00118	0.00122	0.00078	0.00078	0.00081
19	0.00011	0.00021	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.00088	0.00116	0.00041	0.00154	0.00562	0.00822	0.00031	0.00389	0.00123
21	0.00017	0.0	0.00048	0.00106	0.00053	0.00041	0.0	0.00010	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00066	0.0
24	0.00252	0.00312	0.00443	0.00186	0.00484	0.00143	0.00126	0.00064	0.00098
25	0.00080	0.00079	0.01359	0.00430	0.00951	0.00602	0.00538	0.00235	0.00094
26	0.00149	0.00102	0.00306	0.00054	0.00084	0.00078	0.00166	0.00050	0.00045
27	0.00906	0.00904	0.0	0.00267	0.00362	0.01144	0.00006	0.00106	0.00087
28	0.00025	0.01270	0.00099	0.00029	0.00447	0.00083	0.00073	0.00016	0.00051
29	0.00186	0.00109	0.00235	0.00029	0.00058	0.00019	0.00050	0.00064	0.00055
30	0.00079	0.00083	0.01730	0.00288	0.00555	0.00181	0.00134	0.00422	0.00156
31	0.00769	0.00471	0.00274	0.00577	0.00709	0.00527	0.00448	0.00422	0.00473
32	0.00322	0.00164	0.01481	0.00137	0.00632	0.00711	0.00550	0.03574	0.01599
33	0.0	0.0	0.00038	0.00014	0.00013	0.00011	0.00016	0.00116	0.00014
34	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001
35	0.00006	0.00003	0.00002	0.00407	0.00109	0.00030	0.00060	0.00012	0.00003
36	0.01543	0.00483	0.00200	0.00604	0.00836	0.00811	0.00735	0.00586	0.00598
37	0.23078	0.00280	0.44647	0.24820	0.20118	0.14812	0.11448	0.15631	0.16448
38	0.01436	0.33581	0.00876	0.07422	0.06969	0.06618	0.03629	0.00807	0.00769
39	0.0	0.0	0.00158	0.0	0.0	0.0	0.0	0.0	0.0
40	0.0	0.0	0.0	0.02041	0.00028	0.0	0.00005	0.0	0.00997
41	0.00598	0.00966	0.00941	0.01589	0.03016	0.01590	0.02137	0.03097	0.01083
42	0.01575	0.01034	0.00333	0.02762	0.01840	0.04500	0.00284	0.00450	0.01012
43	0.0	0.0	0.0	0.00012	0.00166	0.0	0.10320	0.05155	0.02717
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.03993	0.00092
45	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00103	0.06021
46	0.0	0.0	0.0	0.0	0.0	0.0	0.00139	0.0	0.0
47	0.00588	0.00732	0.01247	0.00512	0.00819	0.02773	0.01821	0.01686	0.01476
48	0.00275	0.0	0.0	0.00001	0.00028	0.00017	0.0	0.0	0.0
49	0.00255	0.00308	0.00692	0.00889	0.00002	0.00227	0.02392	0.05941	0.05421
50	0.00741	0.00370	0.00432	0.00347	0.00343	0.00206	0.03390	0.01541	0.00452
51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.0	0.0	0.00184	0.0	0.0	0.0	0.0	0.0
53	0.00425	0.00115	0.00105	0.01066	0.00196	0.00404	0.00978	0.00568	0.01160
54	0.0	0.0	0.0	0.00005	0.00025	0.0	0.0	0.00191	0.0
55	0.00021	0.00446	0.00041	0.00102	0.00174	0.00078	0.00049	0.00073	0.00053
56	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
57	0.0	0.0	0.0	0.0	0.00028	0.00049	0.0	0.0	0.0
58	0.00001	0.00001	0.00001	0.00003	0.00001	0.00002	0.01882	0.00783	0.00166
59	0.0	0.0	0.0	0.0	0.0	0.0	0.00149	0.00672	0.00041
60	0.0	0.0	0.0	0.0	0.00030	0.0	0.00099	0.0	0.0
61	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.00027	0.00023	0.00020	0.00875	0.00040	0.00057	0.00087	0.00133	0.00032
63	0.00007	0.00006	0.00006	0.00009	0.00011	0.00011	0.00007	0.00007	0.00007
64	0.00061	0.00076	0.00073	0.00070	0.00052	0.00067	0.00147	0.00063	0.00058
65	0.05131	0.02244	0.02573	0.01937	0.01709	0.01560	0.01394	0.01725	0.01542
66	0.00356	0.00313	0.00102	0.00438	0.00286	0.00320	0.00287	0.00267	0.00345
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.02378	0.02440	0.00675	0.00678	0.00997	0.00865	0.00484	0.00590	0.00657
69	0.03569	0.03518	0.03933	0.03941	0.03247	0.03980	0.03023	0.04307	0.03793
70	0.00748	0.00681	0.00630	0.00869	0.00882	0.00783	0.00666	0.00688	0.00755
71	0.00391	0.00443	0.00419	0.00673	0.00935	0.00645	0.00509	0.00568	0.00587
72	0.00113	0.00084	0.00087	0.00136	0.00164	0.00171	0.00135	0.00113	0.00113
73	0.00882	0.00880	0.00947	0.01241	0.01022	0.01714	0.01296	0.02837	0.01485
74	0.00389	0.00056	0.0	0.00027	0.0	0.0	0.00614	0.0	0.0
75	0.00035	0.00066	0.00041	0.00225	0.00090	0.00116	0.00032	0.00110	0.00092
76	0.00003	0.00002	0.00002	0.00003	0.00004	0.00004	0.00003	0.00003	0.00003
77	0.00104	0.00104	0.00103	0.00108	0.00110	0.00111	0.00107	0.00107	0.00107
78	0.00080	0.00053	0.00071	0.00110	0.00114	0.00116	0.00111	0.00204	0.00084
79	0.00071	0.00033	0.00010	0.00024	0.00030	0.00028	0.00016	0.00033	0.00016

COL. ROW	46	47	48	49	50	51	52	53	54
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.00019	0.0	0.0	0.0	0.0	0.0	0.00069	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.00002	0.00023	0.00014	0.0	0.00038	0.00036	0.00035
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.00280	0.00003	0.0	0.0	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00039	0.00251	0.00107	0.00134	0.00443	0.00076	0.00139	0.00144	0.00066
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.00179	0.0	0.00139	0.00053	0.0	0.00006	0.00013	0.00032	0.00266
17	0.0	0.0	0.0	0.0	0.0	0.00000	0.0	0.00034	0.00007
18	0.00087	0.00128	0.00113	0.00108	0.00141	0.00089	0.00082	0.00099	0.00074
19	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.00092	0.00164	0.00556	0.00159	0.0	0.00056	0.00301	0.00139	0.00166
21	0.00002	0.00000	0.00016	0.00007	0.0	0.00035	0.00568	0.00003	0.00599
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.00077	0.0	0.00195	0.00305	0.00009	0.00661	0.00447	0.00860	0.00158
25	0.00043	0.00033	0.00031	0.00190	0.0	0.00182	0.00963	0.00427	0.01009
26	0.00067	0.00007	0.00019	0.00062	0.00016	0.00012	0.00018	0.00129	0.00067
27	0.0	0.00114	0.00174	0.00158	0.0	0.00036	0.00740	0.00386	0.00535
28	0.00059	0.00039	0.00081	0.00031	0.0	0.00121	0.00115	0.00753	0.00273
29	0.00082	0.00047	0.00075	0.00033	0.00062	0.00022	0.00027	0.00012	0.00019
30	0.00301	0.00006	0.00036	0.00077	0.0	0.00102	0.00575	0.00315	0.00607
31	0.00400	0.00594	0.00631	0.00427	0.01088	0.00157	0.00340	0.00342	0.00134
32	0.02255	0.00042	0.00903	0.00460	0.00124	0.01078	0.01334	0.00773	0.03430
33	0.00012	0.00009	0.00136	0.00028	0.00034	0.00011	0.00016	0.00012	0.00026
34	0.00001	0.00001	0.00001	0.00001	0.00001	0.00002	0.00001	0.00002	0.00132
35	0.00036	0.00016	0.00004	0.00002	0.00003	0.00027	0.00191	0.00097	0.00131
36	0.00453	0.00804	0.00442	0.00875	0.01673	0.00233	0.00688	0.00845	0.00819
37	0.12929	0.07946	0.09745	0.11513	0.08004	0.01962	0.07913	0.06299	0.08000
38	0.01395	0.02285	0.04546	0.02748	0.07707	0.01957	0.05857	0.06909	0.04421
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
40	0.00893	0.00117	0.00856	0.00674	0.0	0.0	0.00313	0.00247	0.00267
41	0.01498	0.00892	0.01113	0.01188	0.00216	0.01151	0.04017	0.01597	0.05064
42	0.01210	0.00445	0.01317	0.01492	0.01253	0.00553	0.02542	0.00730	0.03272
43	0.00907	0.0	0.00010	0.00335	0.0	0.0	0.00129	0.00172	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.00152	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
46	0.04480	0.00107	0.00002	0.0	0.0	0.0	0.0	0.0	0.0
47	0.01371	0.06715	0.01289	0.01677	0.01612	0.00969	0.00291	0.01121	0.00784
48	0.0	0.00083	0.05561	0.00004	0.0	0.0	0.0	0.0	0.00002
49	0.05638	0.03195	0.04732	0.07909	0.00217	0.00623	0.00616	0.00899	0.00933
50	0.01679	0.00169	0.00369	0.00402	0.07265	0.00219	0.00102	0.00174	0.00102
51	0.00011	0.0	0.00080	0.0	0.0	0.09054	0.0	0.0	0.0
52	0.0	0.0	0.00002	0.0	0.0	0.0	0.05442	0.00013	0.02010
53	0.05586	0.02759	0.03762	0.04092	0.00508	0.02140	0.09942	0.07291	0.03799
54	0.0	0.0	0.0	0.0	0.0	0.0	0.00007	0.0	0.01125
55	0.00136	0.00025	0.00027	0.00078	0.00082	0.00421	0.00747	0.00874	0.00760
56	0.0	0.0	0.0	0.0	0.0	0.00039	0.0	0.00002	0.0
57	0.0	0.0	0.0	0.00019	0.0	0.04336	0.00007	0.02211	0.0
58	0.00273	0.00001	0.00002	0.00021	0.00002	0.00000	0.00002	0.00001	[illegible]
59	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
60	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.00039	0.00060	0.00134	0.00493	0.00074	0.00047	0.01418	0.00124	0.03108
63	0.00008	0.00015	0.00011	0.00014	0.00013	0.00008	0.00008	0.00009	0.00007
64	0.00055	0.00063	0.00058	0.00032	0.00034	0.00147	0.00131	0.00048	0.00076
65	0.01423	0.00923	0.01251	0.01518	0.01050	0.00864	0.01730	0.01298	0.01661
66	0.00410	0.00963	0.01076	0.01035	0.00733	0.00442	0.00401	0.00449	0.00435
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.00470	0.00721	0.00619	0.00658	0.00847	0.00347	0.00576	0.00748	0.00634
69	0.05043	0.03323	0.04291	0.05221	0.03314	0.05036	0.06277	0.03303	0.04424
70	0.00922	0.00908	0.00788	0.00710	0.00882	0.00538	0.00930	0.00485	0.00343
71	0.01108	0.02066	0.01096	0.00813	0.01599	0.00809	0.01492	0.00849	0.00551
72	0.00126	0.00185	0.00164	0.00144	0.00205	0.00124	0.00115	0.00139	0.00104
73	0.01793	0.01225	0.01343	0.01347	0.01106	0.02706	0.01121	0.01214	0.08972
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.00080	0.00081	0.00164	0.00083	0.00133	0.00027	0.00128	0.00054	0.00019
76	0.00004	0.00005	0.00004	0.00004	0.00005	0.00003	0.00003	0.00004	0.00002
77	0.00106	0.00111	0.00110	0.00110	0.00113	0.00091	0.00108	0.00110	0.00105
78	0.00147	0.00093	0.00106	0.00116	0.00131	0.00161	0.00123	0.00270	0.00219
79	0.00010	0.00014	0.00021	0.00018	0.00019	0.00010	0.00019	0.00012	0.00020

COL. / ROW	55	56	57	58	59	60	61	62	63
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00141	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.00141	0.0	0.0	0.0	0.0	0.0
6	0.00155	0.0	0.0	0.00116	0.0	0.0	0.0	0.00049	0.0
7	0.0	0.00031	0.00018	0.0	0.00066	0.00019	0.00072	0.0	0.00169
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.00019	0.00001	0.0	0.00007	0.00004	0.0
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00006	0.00148	0.00084	0.00007	0.00315	0.00190	0.00038	0.00018	0.00036
13	0.0	0.00018	0.0	0.0	0.0	0.01666	0.0	0.0	0.0
14	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00376	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.00026	0.0	0.00037	0.00224	0.00016	0.00063	0.00900	0.00056
17	0.0	0.0	0.0	0.0	0.00391	0.00052	0.00074	0.00205	0.00032
18	0.00104	0.00080	0.00148	0.00104	0.00046	0.00095	0.00112	0.00168	0.00091
19	0.0	0.0	0.0	0.0	0.00665	0.0	0.00043	0.0	0.0
20	0.00076	0.00192	0.00029	0.0	0.00049	0.00169	0.02446	0.00039	0.00007
21	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00034	0.0
22	0.0	0.02507	0.00595	0.0	0.0	0.0	0.00678	0.0	0.0
23	0.0	0.0	0.0	0.0	0.00020	0.00135	0.00398	0.0	0.0
24	0.00296	0.00475	0.00542	0.00143	0.00369	0.00042	0.00174	0.00504	0.03414
25	0.01882	0.00555	0.00614	0.00787	0.00109	0.00079	0.00035	0.00903	0.00719
26	0.00034	0.00137	0.00059	0.00024	0.00054	0.00085	0.00051	0.00064	0.00027
27	0.00583	0.00109	0.01267	0.02391	0.00181	0.00131	0.00234	0.00299	0.06416
28	0.01982	0.00741	0.00847	0.00623	0.00117	0.00072	0.00752	0.00278	0.00074
29	0.00018	0.00036	0.00006	0.00008	0.00082	0.00043	0.00063	0.00020	0.00007
30	0.00508	0.00047	0.00070	0.00001	0.00394	0.00087	0.00826	0.00060	0.00021
31	0.00192	0.00132	0.00176	0.00155	0.00211	0.00254	0.00454	0.00151	0.00292
32	0.01632	0.01050	0.00734	0.05110	0.02764	0.00662	0.00921	0.01286	0.00782
33	0.00018	0.00019	0.00006	0.00008	0.00029	0.0	0.00045	0.00084	0.00007
34	0.00001	0.00001	0.00002	0.00001	0.00000	0.00001	0.00001	0.00070	0.00052
35	0.03080	0.00486	0.03522	0.00072	0.01037	0.00013	0.00528	0.00366	0.01112
36	0.00619	0.00336	0.00727	0.01049	0.00274	0.00374	0.00854	0.00401	0.00301
37	0.06689	0.00989	0.02091	0.03242	0.08882	0.03392	0.12005	0.02199	0.00631
38	0.25000	0.02081	0.04920	0.11192	0.01153	0.02853	0.01898	0.04497	0.02893
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00276	0.0
40	0.0	0.00051	0.00009	0.0	0.00032	0.00007	0.04307	0.00084	0.0
41	0.02743	0.01986	0.02577	0.02641	0.03101	0.02046	0.00492	0.02043	0.00639
42	0.01624	0.01272	0.01451	0.00423	0.03623	0.01004	0.02035	0.01287	0.01067
43	0.0	0.0	0.0	0.0	0.00273	0.00015	0.02749	0.0	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.0	0.0	0.0	0.0	0.00005	0.0	0.00494	0.0	0.0
46	0.0	0.0	0.0	0.0	0.0	0.0	0.00359	0.0	0.0
47	0.00565	0.00455	0.00634	0.01494	0.01134	0.01987	0.00508	0.01318	0.00459
48	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
49	0.00011	0.00034	0.00069	0.01857	0.00470	0.01027	0.01547	0.00447	0.0
50	0.00171	0.00099	0.00113	0.00239	0.00596	0.01048	0.00368	0.00583	0.0
51	0.0	0.0	0.0	0.0	0.0	0.00007	0.0	0.00666	0.0
52	0.0	0.00067	0.0	0.0	0.00121	0.00008	0.00248	0.00013	0.0
53	0.01669	0.00994	0.00456	0.00892	0.00127	0.00299	0.03409	0.02340	0.00927
54	0.0	0.0	0.0	0.0	0.0	0.0	0.00527	0.0	0.0
55	0.04277	0.01112	0.00465	0.01913	0.00404	0.00130	0.00333	0.00347	0.00275
56	0.0	0.05860	0.00115	0.0	0.00500	0.02581	0.00164	0.00509	0.0
57	0.0	0.18339	0.06755	0.00774	0.00076	0.00622	0.00010	0.03103	0.0
58	0.02073	0.00030	0.0	0.04150	0.01447	0.00380	0.00125	0.00008	0.00001
59	0.0	0.0	0.0	0.00496	0.30122	0.00342	0.00805	0.0	0.0
60	0.0	0.0	0.0	0.0	0.0	0.20202	0.0	0.00159	0.0
61	0.0	0.0	0.0	0.0	0.00013	0.00008	0.06978	0.0	0.0
62	0.00038	0.00201	0.00106	0.00077	0.00432	0.01343	0.00143	0.06576	0.00069
63	0.00010	0.00007	0.00014	0.00010	0.00004	0.00213	0.00011	0.00119	0.05858
64	0.00314	0.00049	0.00098	0.00060	0.00087	0.00066	0.00224	0.00446	0.00046
65	0.01423	0.01225	0.01167	0.01437	0.01889	0.00938	0.01809	0.01104	0.01517
66	0.00282	0.00359	0.00314	0.00364	0.00210	0.00473	0.00286	0.00509	0.00327
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.00564	0.00287	0.00821	0.00688	0.00480	0.00578	0.00656	0.00439	0.00392
69	0.06683	0.03736	0.06052	0.03784	0.03048	0.01916	0.04935	0.04332	0.03781
70	0.00463	0.00359	0.00530	0.00451	0.00385	0.00316	0.00527	0.00579	0.00577
71	0.01003	0.00646	0.02228	0.00816	0.00280	0.00603	0.00470	0.01145	0.01385
72	0.00145	0.00112	0.00212	0.00143	0.00066	0.0	0.00160	0.00153	0.00132
73	0.01409	0.02446	0.01122	0.02390	0.02545	0.00366	0.00897	0.01696	0.06105
74	0.0	0.00035	0.00094	0.0	0.00053	0.00100	0.0	0.0	0.00137
75	0.00033	0.00024	0.00002	0.00052	0.00036	0.00014	0.00095	0.00029	0.00060
76	0.00003	0.00003	0.00005	0.00003	0.00002	0.0	0.00004	0.00004	0.00004
77	0.00109	0.00107	0.00114	0.00107	0.00102	0.00096	0.00109	0.00111	0.00109
78	0.00159	0.00255	0.00364	0.00255	0.00172	0.00100	0.00079	0.00130	0.00226
79	0.00020	0.00010	0.00020	0.00025	0.00018	0.00017	0.00023	0.00016	0.00011

COL. / ROW	64	65	66	67	68	69	70	71	72
1	0.0	0.00005	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.00146	0.00024	0.0	0.0	0.0	0.0	0.0	0.00036	0.0
3	0.00057	0.00004	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.00166	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.00015	0.00082	0.0	0.0	0.03163	0.0	0.00023	0.0	0.0
8	0.0	0.0	0.0	0.0	0.06332	0.0	0.0	0.0	0.0
9	0.00007	0.00004	0.0	0.0	0.0	0.0	0.0	0.00007	0.0
10	0.00001	0.00003	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00309	0.03834	0.03153	0.00517	0.03207	0.00841	0.00453	0.10672	0.00288
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.00137	0.00307	0.0	0.0	0.0	0.00002	0.0	0.0	0.00108
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.01951	0.00021	0.00007	0.0	0.0	0.0	0.0	0.0	0.01029
17	0.00671	0.00048	0.00038	0.00113	0.0	0.00018	0.00080	0.0	0.00191
18	0.00082	0.00013	0.0	0.0	0.0	0.0	0.00001	0.0	0.00794
19	0.00036	0.00046	0.00055	0.00166	0.0	0.00024	0.00118	0.0	0.01066
20	0.01831	0.00028	0.00002	0.0	0.0	0.00005	0.0	0.0	0.00041
21	0.00036	0.00053	0.0	0.0	0.0	0.00097	0.0	0.0	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00091
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00028
24	0.02198	0.00103	0.0	0.00535	0.00052	0.00469	0.00385	0.0	0.01196
25	0.04001	0.00026	0.0	0.00128	0.0	0.00376	0.00092	0.0	0.00155
26	0.00241	0.00231	0.01186	0.00630	0.00030	0.00189	0.01492	0.00001	0.00053
27	0.00751	0.00109	0.00003	0.0	0.00001	0.0	0.0	0.00031	0.00806
28	0.02406	0.00000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
29	0.00097	0.00035	0.00010	0.00003	0.00002	0.0	0.00042	0.0	0.01455
30	0.00782	0.00122	0.0	0.0	0.0	0.0	0.0	0.0	0.00001
31	0.00334	0.04662	0.00175	0.00130	0.01406	0.00746	0.00354	0.00444	0.01181
32	0.02360	0.00790	0.00067	0.00055	0.00046	0.00170	0.00201	0.00060	0.00566
33	0.00656	0.00010	0.0	0.0	0.0	0.0	0.0	0.0	0.0
34	0.00510	0.00001	0.00003	0.00003	0.00001	0.00004	0.00004	0.00000	0.00056
35	0.00527	0.00017	0.0	0.0	0.0	0.00110	0.0	0.0	0.00026
36	0.00207	0.00012	0.0	0.0	0.00134	0.00045	0.0	0.0	0.00425
37	0.02863	0.00116	0.0	0.0	0.00016	0.0	0.0	0.0	0.0
38	0.04920	0.00149	0.00246	0.0	0.0	0.0	0.0	0.0	0.00052
39	0.0	0.0	0.0	0.0	0.0	0.00004	0.0	0.0	0.0
40	0.0	0.00002	0.0	0.0	0.0	0.0	0.0	0.0	0.0
41	0.00833	0.00047	0.0	0.0	0.0	0.00021	0.0	0.0	0.0
42	0.01619	0.00126	0.00031	0.0	0.00920	0.00010	0.0	0.0	0.00229
43	0.0	0.00249	0.0	0.0	0.0	0.0	0.0	0.0	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
46	0.0	0.00043	0.0	0.0	0.0	0.0	0.0	0.0	0.0
47	0.00043	0.00072	0.00001	0.0	0.00013	0.0	0.0	0.0	0.0
48	[illegible]	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00005
49	0.00067	0.00044	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	0.00721	0.00021	0.0	0.0	0.0	0.00014	0.0	0.0	0.0
51	0.0	0.00008	0.0	0.0	0.00046	0.00008	0.00018	0.0	0.0
52	0.00030	0.00007	0.0	0.0	0.0	0.00005	0.0	0.0	0.00328
53	0.00573	0.00095	0.00004	0.0	0.00057	0.0	0.0	0.0	0.00054
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00970
55	0.00281	0.00022	0.0	0.0	0.00015	0.00004	0.0	0.0	0.00061
56	0.0	0.00054	0.01244	0.02265	0.00021	0.00003	0.0	0.0	0.00082
57	0.00173	0.00077	0.00057	0.0	0.0	0.0	0.0	0.0	0.01913
58	0.00002	0.00224	0.00048	0.0	0.00002	0.00024	0.00007	[illegible]	[illegible]
59	0.0	0.00270	0.0	0.0	0.0	0.00177	0.0	0.0	0.0
60	0.00019	0.00502	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.0	0.00931	0.00051	0.0	0.00007	0.0	0.00036	0.0	0.00074
62	0.00018	0.00079	0.00001	0.0	0.00000	0.0	0.0	0.0	0.00575
63	0.0	0.00000	0.00000	0.0	0.0	0.0	0.0	0.0	0.00868
64	0.05801	0.00147	0.00097	0.00575	0.00036	0.00048	0.00085	0.0	0.02121
65	0.01615	0.06468	0.00177	0.00165	0.02141	0.00379	0.00890	0.00048	0.00724
66	0.00580	0.00833	0.00893	0.03955	0.00272	0.01098	0.01560	[illegible]	0.00530
67	0.0	0.0	0.0	0.00268	0.0	0.0	0.0	0.0	0.0
68	0.00525	0.00452	0.00613	0.00405	0.19677	0.02078	0.00474	0.00379	0.01931
69	0.06305	0.03083	0.00527	0.01177	0.01397	0.01715	0.00969	0.00303	0.04270
70	0.00871	0.02155	0.00566	0.00812	0.00644	0.01713	0.20413	0.02344	0.01624
71	0.01571	0.03141	0.01641	0.05034	0.00324	0.05475	0.07799	0.02293	0.04643
72	0.00191	0.0	0.0	0.0	0.0	0.00244	0.0	0.0	0.02908
73	0.02455	0.01528	0.01753	0.04018	0.01312	0.05157	0.03882	0.01281	0.02589
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.00155	0.02519	0.00163	0.00109	0.00144	0.00896	0.00315	0.00081	0.00949
76	0.00004	0.00077	0.00031	0.19413	0.0	0.00104	0.00019	0.0	0.0
77	0.00112	0.00097	0.00093	0.00097	0.00096	0.00103	0.00532	0.00023	0.00096
78	0.00191	0.00132	0.00414	0.00059	0.00290	0.00450	0.01237	0.00442	0.00058
79	0.00026	0.00172	0.00041	0.00009	0.00035	0.00055	0.00028	0.00004	0.00153

COL. ROW	73	74	75	76	77	78	79
1	0.0	0.0	0.0	0.00193	0.00020	0.00042	0.0
2	0.0	0.0	0.0	0.0	0.00023	0.15143	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.00055	0.0	0.0	0.00012
5	0.0	0.0	0.0	0.0	0.0	0.00043	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.00182	0.0	0.00133	0.0	0.00001	0.01140	0.01657
8	0.0	0.0	0.0	0.0	0.0	0.0	0.00417
9	0.0	0.0	0.0	0.0	0.0	0.00010	0.0
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00134	0.0	0.01371	0.02242	0.02997	0.00345	0.24517
13	0.0	0.0	0.0	0.0	0.0	0.0	0.00000
14	0.0	0.0	0.0	0.0	0.00747	0.06319	0.00011
15	0.0	0.0	0.0	0.0	0.00001	0.0	0.0
16	0.0	0.0	0.0	0.0	0.00011	0.0	0.0
17	0.00085	0.00327	0.00206	0.00057	0.00100	0.0	0.00045
18	0.00026	0.00381	0.00014	0.0	0.00168	0.0	0.00035
19	0.00132	0.00481	0.00232	0.00097	0.00151	0.00049	0.0
20	0.0	0.0	0.0	0.0	0.00014	0.0	0.00001
21	0.0	0.0	0.0	0.0	0.0	0.0	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.00400	0.01268	0.00041	0.00060	0.00373	0.00660	0.00040
25	0.00049	0.00303	0.00012	0.00014	0.00089	0.00158	0.0
26	0.00400	0.00038	0.00125	0.00353	0.01420	0.00920	0.00251
27	0.00084	0.0	0.00001	0.0	0.00020	0.0	0.00497
28	0.0	0.0	0.0	0.0	0.0	0.0	0.00001
29	0.00169	0.01049	0.00118	0.00017	0.02591	0.0	0.00060
30	0.0	0.0	0.00775	0.0	0.0	0.0	0.00002
31	0.00659	0.00060	0.00345	0.00066	0.00309	0.00150	0.00844
32	0.00398	0.03210	0.03445	0.00031	0.00283	0.00026	0.00114
33	0.0	0.0	0.0	0.0	0.0	0.0	0.0
34	0.00003	0.0	0.00001	0.00174	0.00011	0.00018	0.0
35	0.0	0.0	0.01230	0.0	0.00028	0.0	0.0
36	0.00001	0.0	0.00498	0.0	0.00000	0.00287	0.00013
37	0.0	0.0	0.0	0.0	0.0	0.0	0.00047
38	0.00096	0.0	0.0	0.0	0.0	0.0	0.0
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0
40	0.0	0.0	0.0	0.0	0.0	0.0	0.0
41	0.0	0.0	0.0	0.0	0.00089	0.0	0.0
42	0.00005	0.0	0.01439	0.0	0.00002	0.00072	0.00449
43	0.00622	0.0	0.0	0.0	0.0	0.0	0.0
44	0.00825	0.0	0.0	0.0	0.0	0.0	0.0
45	0.00108	0.0	0.0	0.0	0.0	0.0	0.0
46	0.0	0.0	0.0	0.0	0.0	0.0	0.0
47	0.00112	0.00174	0.00012	0.0	0.0	0.0	0.0
48	0.0	0.0	0.0	0.0	0.0	0.0	0.0
49	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	0.00002	0.0	0.01329	0.0	0.00001	0.00018	0.00022
51	0.00488	0.0	0.0	0.0	0.00013	0.0	0.0
52	0.00523	0.00282	0.0	0.0	0.0	0.0	0.0
53	0.00001	0.0	0.00077	0.0	0.0	0.0	0.0
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0
55	0.00001	0.0	0.00420	0.0	0.00000	0.00005	0.00006
56	0.0	0.0	0.0	0.0	0.00049	0.0	0.0
57	0.0	0.0	0.0	0.0	0.0	0.0	0.0
58	0.00028	0.0	0.01453	0.0	0.00051	0.00012	0.00020
59	0.00030	0.0	0.14335	0.0	0.00008	0.00228	0.00257
60	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.00111	0.01069	0.00121	0.00017	0.00086	0.0	0.0
62	0.0	0.0	0.00203	0.0	0.01144	0.0	0.0
63	0.01167	0.0	0.0	0.00439	0.00318	0.0	0.0
64	0.00446	0.00923	0.00023	0.01544	0.00141	0.0	0.00007
65	0.00685	0.00376	0.00941	0.00401	0.00528	0.17317	0.01455
66	0.01655	0.00374	0.00700	0.00553	0.00838	0.00276	0.00523
67	0.00021	0.0	0.0	0.0	0.0	0.0	0.0
68	0.01496	0.0	0.01917	0.00753	0.01840	0.01773	0.07865
69	0.02564	0.02176	0.08507	0.01254	0.01850	0.01509	0.00880
70	0.01787	0.00936	0.02625	0.02307	0.01165	0.00124	0.00941
71	0.05442	0.02178	0.03929	0.05079	0.06858	0.00991	0.01487
72	0.0	0.0	0.0	0.0	0.00485	0.0	0.00155
73	0.03333	0.03743	0.01950	0.03792	0.02522	0.01355	0.01425
74	0.0	0.0	0.0	0.0	0.00176	0.00032	0.0
75	0.00713	0.0	0.01687	0.0	0.00215	0.00866	0.00101
76	0.0	0.00936	0.0	0.23640	0.00367	0.0	0.0
77	0.00022	0.00097	0.00097	0.00096	0.01304	0.0	0.00005
78	0.03994	0.0	0.00045	0.00056	0.00070	0.00154	0.00085
79	0.00044	0.0	0.00008	0.00022	0.00054	0.00021	0.00010

TABLE B-9

"MAKE" MATRIX, UNITED STATES ECONOMY, 1958

(thousands of 1958 dollars)

$\dot{X} = [\dot{x}_{kl}]$ k, l = 1, ..., 79.

(refer to Chapter II, Section I)

COL. ROW	1	2	3	4	5	6	7	8	9
1	23964864.	0.	74922.	0.	0.	0.	0.	0.	0.
2	0.	20669408.	151615.	545207.	0.	0.	0.	0.	0.
3	0.	0.	905486.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	1012771.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	746480.	12728.	0.	0.	403.
6	0.	0.	0.	0.	18896.	987894.	0.	0.	963.
7	0.	0.	0.	0.	0.	0.	2729914.	0.	1000.
8	0.	0.	0.	0.	0.	0.	0.	9133814.	0.
9	0.	0.	0.	0.	0.	73.	435.	0.	1351992.
10	0.	0.	0.	0.	0.	1210.	0.	0.	673.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	0.	0.	0.	0.	0.	0.	0.	0.	0.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	6438.	0.	0.	0.	0.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	0.	0.	0.	0.	0.	0.	0.	0.
17	0.	0.	0.	0.	0.	0.	0.	0.	0.
18	0.	0.	0.	0.	0.	0.	0.	0.	0.
19	0.	0.	0.	0.	0.	0.	0.	0.	0.
20	0.	0.	0.	0.	0.	0.	0.	0.	0.
21	0.	0.	0.	0.	0.	0.	0.	0.	0.
22	0.	0.	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.	0.	0.
24	0.	0.	0.	0.	0.	0.	0.	0.	0.
25	0.	0.	0.	0.	0.	0.	0.	0.	0.
26	0.	0.	0.	0.	0.	0.	0.	0.	0.
27	0.	0.	0.	0.	0.	0.	0.	0.	0.
28	0.	0.	0.	0.	0.	0.	0.	0.	0.
29	0.	0.	0.	0.	0.	0.	0.	0.	0.
30	0.	0.	0.	0.	0.	0.	0.	0.	0.
31	0.	0.	0.	0.	0.	0.	0.	0.	0.
32	0.	0.	0.	0.	0.	0.	0.	0.	0.
33	0.	0.	0.	0.	0.	0.	0.	0.	0.
34	0.	0.	0.	0.	0.	0.	0.	0.	0.
35	0.	0.	0.	0.	0.	0.	0.	0.	0.
36	0.	0.	0.	0.	0.	0.	0.	0.	100159.
37	0.	0.	0.	0.	0.	0.	0.	0.	0.
38	0.	0.	0.	0.	0.	0.	0.	0.	0.
39	0.	0.	0.	0.	0.	0.	0.	0.	0.
40	0.	0.	0.	0.	0.	0.	0.	0.	0.
41	0.	0.	0.	0.	0.	0.	0.	0.	0.
42	0.	0.	0.	0.	0.	0.	0.	0.	0.
43	0.	0.	0.	0.	0.	0.	0.	0.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	0.	0.	0.	0.	0.	0.	0.	0.	0.
46	0.	0.	0.	0.	0.	0.	0.	0.	0.
47	0.	0.	0.	0.	0.	0.	0.	0.	0.
48	0.	0.	0.	0.	0.	0.	0.	0.	0.
49	0.	0.	0.	0.	0.	0.	0.	0.	0.
50	0.	0.	0.	0.	0.	0.	0.	0.	0.
51	0.	0.	0.	0.	0.	0.	0.	0.	0.
52	0.	0.	0.	0.	0.	0.	0.	0.	0.
53	0.	0.	0.	0.	0.	0.	0.	0.	0.
54	0.	0.	0.	0.	0.	0.	0.	0.	0.
55	0.	0.	0.	0.	0.	0.	0.	0.	0.
56	0.	0.	0.	0.	0.	0.	0.	0.	0.
57	0.	0.	0.	0.	0.	0.	0.	0.	0.
58	0.	0.	0.	0.	0.	0.	0.	0.	0.
59	0.	0.	0.	0.	0.	0.	0.	0.	0.
60	0.	0.	0.	0.	0.	0.	0.	0.	0.
61	0.	0.	0.	0.	0.	0.	0.	0.	0.
62	0.	0.	0.	0.	0.	0.	0.	0.	0.
63	0.	0.	0.	0.	0.	0.	0.	0.	0.
64	0.	0.	0.	0.	0.	0.	0.	0.	0.
65	0.	0.	0.	0.	0.	0.	0.	0.	0.
66	0.	0.	0.	0.	0.	0.	0.	0.	0.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	0.	0.	0.	0.	0.	0.	0.	0.	0.
69	0.	0.	0.	0.	0.	0.	0.	0.	0.
70	0.	0.	0.	0.	0.	0.	0.	0.	0.
71	0.	0.	0.	0.	0.	0.	0.	0.	0.
72	0.	0.	0.	0.	0.	0.	0.	0.	0.
73	0.	0.	0.	0.	0.	0.	0.	0.	0.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	0.	0.	0.	0.	0.	0.	0.	0.	0.
76	0.	0.	0.	0.	0.	0.	0.	0.	0.
77	0.	0.	0.	0.	0.	0.	0.	0.	0.
78	0.	0.	0.	0.	0.	0.	0.	0.	0.
79	0.	0.	0.	0.	0.	0.	0.	0.	0.

COL. ROW	1C	11	12	13	14	15	16	17	18
1	0.	0.	0.	0.	1323928.	0.	0.	0.	0.
2	0.	0.	0.	0.	86908.	0.	0.	0.	0.
3	0.	0.	0.	0.	C.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	352.	0.	0.	0.	0.	0.	0.	0.	0.
6	300.	0.	0.	0.	0.	0.	0.	0.	0.
7	0.	0.	0.	C.	0.	0.	0.	0.	0.
8	1200.	0.	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	C.	0.	0.	0.	0.	0.	0.
10	308430.	0.	0.	0.	0.	C.	0.	0.	0.
11	0.	52416000.	0.	0.	0.	0.	0.	0.	0.
12	0.	0.	16867664.	0.	C.	0.	0.	0.	0.
13	0.	0.	0.	2301017.	0.	C.	0.	0.	0.
14	0.	0.	C.	0.	60620480.	100.	0.	11.	0.
15	0.	0.	C.	C.	521.	5767103.	0.	0.	0.
16	0.	0.	0.	0.	5390.	0.	9932830.	64500.	25000.
17	0.	0.	0.	0.	58.	0.	99000.	1853025.	750.
18	0.	0.	0.	0.	1000.	0.	16500.	7000.	14032219.
19	0.	0.	0.	0.	0.	0.	28000.	12000.	12000.
20	0.	0.	0.	C.	63.	0.	80.	11.	0.
21	0.	0.	0.	0.	68.	0.	0.	0.	59.
22	0.	0.	0.	71.	0.	0.	172.	6300.	54.
23	0.	0.	0.	0.	C.	0.	46.	0.	0.
24	0.	0.	0.	0.	68.	0.	160.	11500.	59.
25	0.	0.	C.	0.	0.	0.	0.	11.	0.
26	0.	0.	0.	0.	0.	195.	167.	0.	51.
27	13060.	0.	C.	0.	33000.	0.	0.	0.	0.
28	0.	0.	C.	0.	3000.	0.	1500.	10000.	0.
29	0.	0.	0.	5.	53500.	0.	37.	0.	27.
30	0.	0.	C.	C.	25.	0.	0.	900.	0.
31	0.	0.	0.	0.	0.	0.	0.	0.	0.
32	0.	0.	0.	6.	3000.	0.	3000.	32000.	6100.
33	0.	0.	0.	C.	101.	0.	0.	0.	123.
34	0.	0.	0.	125.	0.	0.	255.	900.	11000.
35	0.	0.	C.	0.	0.	0.	0.	0.	0.
36	0.	0.	0.	0.	500.	5.	1500.	1900.	64.
37	0.	0.	0.	11000.	0.	0.	0.	0.	C.
38	0.	0.	0.	3000.	0.	0.	0.	11.	0.
39	0.	0.	0.	0.	C.	0.	C.	11.	0.
40	C.	0.	C.	9000.	0.	0.	500.	0.	0.
41	0.	0.	C.	2000.	0.	0.	0.	0.	0.
42	0.	0.	0.	2000.	C.	0.	125.	1400.	61.
43	C.	0.	0.	0.	0.	0.	0.	C.	0.
44	0.	0.	0.	5000.	0.	0.	0.	900.	0.
45	0.	0.	0.	2000.	0.	0.	0.	0.	0.
46	0.	0.	C.	65.	0.	0.	0.	11.	C.
47	0.	0.	0.	2000.	0.	0.	0.	11.	0.
48	0.	0.	0.	5379.	58.	0.	0.	0.	33.
49	0.	0.	C.	2000.	73.	0.	0.	0.	0.
50	0.	0.	0.	19.	15.	C.	0.	0.	0.
51	0.	0.	C.	3996.	0.	0.	0.	0.	0.
52	0.	0.	C.	18384.	191.	0.	0.	0.	0.
53	0.	0.	0.	10000.	0.	0.	C.	0.	0.
54	0.	0.	C.	1000.	0.	0.	0.	0.	0.
55	0.	0.	0.	1000.	0.	0.	0.	0.	0.
56	0.	0.	C.	18000.	0.	0.	0.	0.	0.
57	0.	0.	C.	209.	C.	0.	0.	0.	0.
58	C.	0.	0.	1000.	0.	0.	0.	0.	0.
59	0.	0.	0.	20000.	0.	0.	0.	900.	0.
60	0.	0.	0.	349033.	0.	0.	0.	700.	0.
61	0.	0.	0.	16500.	0.	0.	0.	0.	59.
62	0.	0.	0.	17000.	68.	0.	167.	11.	2000.
63	0.	0.	0.	3000.	0.	0.	267.	0.	60.
64	0.	0.	0.	5000.	22.	0.	29.	12.	4000.
65	0.	0.	0.	0.	5766.	0.	0.	0.	0.
66	0.	0.	0.	0.	0.	0.	0.	0.	0.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	0.	0.	0.	0.	0.	0.	0.	0.	0.
69	0.	0.	C.	C.	0.	0.	0.	0.	0.
7C	C.	0.	0.	0.	0.	0.	0.	0.	0.
71	0.	0.	0.	0.	0.	0.	0.	0.	0.
72	0.	0.	0.	0.	0.	0.	0.	0.	0.
73	0.	0.	0.	0.	0.	C.	0.	0.	0.
74	0.	0.	0.	C.	0.	0.	0.	0.	C.
75	0.	0.	0.	0.	0.	0.	0.	0.	0.
76	0.	0.	C.	0.	0.	0.	0.	0.	0.
77	0.	0.	0.	0.	0.	C.	0.	0.	0.
78	0.	0.	0.	0.	0.	0.	0.	0.	0.
79	0.	0.	C.	0.	0.	0.	0.	0.	0.

CCL. ROW	19	20	21	22	23	24	25	26	27
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	202875.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	8070.
6	0.	250.	0.	0.	0.	0.	0.	0.	175.
7	0.	0.	0.	0.	0.	0.	0.	0.	0.
8	0.	0.	0.	0.	0.	0.	0.	0.	0.
9	0.	170.	0.	0.	0.	0.	0.	0.	55.
10	0.	0.	0.	0.	0.	0.	0.	0.	144371.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	0.	0.	0.	0.	0.	0.	0.	0.	0.
13	8.	0.	0.	0.	0.	0.	1300.	5500.	0.
14	0.	92.	0.	5.	100.	1300.	234.	0.	56500.
15	0.	0.	0.	0.	0.	50.	50.	0.	0.
16	337640.	3.	0.	0.	0.	1000.	0.	800.	59.
17	11146.	5.	0.	6000.	3000.	7000.	0.	800.	1400.
18	18500.	9.	7.	7.	0.	7.	7.	7.	0.
19	1730234.	500.	0.	2500.	1250.	24000.	319.	2.	625.
20	18.	7429073.	30000.	27000.	7000.	5500.	3000.	4.	1362.
21	0.	20500.	376967.	9.	78.	8.	2000.	32.	0.
22	6808.	18500.	2700.	3053023.	37500.	179.	0.	0.	0.
23	6030.	3000.	500.	24500.	1246093.	1500.	0.	5000.	30.
24	17000.	927.	0.	0.	1000.	8701023.	26500.	71500.	27000.
25	0.	6.	1000.	0.	0.	49000.	3436165.	20000.	0.
26	18.	68.	0.	68.	810.	58500.	15500.	5965364.	0.
27	18.	1500.	0.	0.	0.	1400.	60.	5.	9348472.
28	0.	361.	0.	0.	0.	317.	1200.	0.	210500.
29	1000.	46.	0.	0.	25.	5.	228.	1000.	138500.
30	0.	0.	0.	0.	0.	1000.	0.	0.	32500.
31	0.	500.	0.	0.	0.	1000.	0.	800.	559438.
32	1800.	1500.	125.	6600.	1500.	11500.	8500.	1000.	19000.
33	16.	0.	0.	0.	0.	0.	0.	0.	80.
34	6000.	500.	0.	1000.	4.	1000.	500.	244.	0.
35	32.	36.	0.	36.	1.	0.	6853.	0.	1.
36	0.	3000.	0.	258.	318.	38000.	0.	0.	5000.
37	800.	0.	150.	150.	1500.	1000.	0.	0.	119698.
38	0.	44.	0.	0.	235.	1000.	500.	0.	74729.
39	0.	0.	21.	21.	0.	500.	14200.	25.	0.
40	18.	3000.	231.	10500.	19000.	397.	0.	0.	2000.
41	18.	5000.	0.	5300.	5300.	2000.	0.	476.	7000.
42	1400.	3000.	125.	9000.	3000.	6500.	289.	10000.	2200.
43	0.	0.	0.	0.	0.	0.	0.	0.	0.
44	0.	262.	0.	0.	1000.	0.	0.	0.	204.
45	0.	7.	0.	0.	500.	0.	0.	398.	5.
46	0.	500.	0.	2000.	65.	0.	0.	0.	59.
47	0.	17.	0.	87.	52.	179.	229.	129.	6000.
48	0.	500.	1000.	5600.	37.	1000.	60.	33.	33.
49	800.	325.	0.	274.	2000.	190.	0.	0.	2000.
50	0.	19.	0.	0.	19.	0.	0.	0.	1000.
51	18.	0.	0.	0.	3030.	1700.	0.	4800.	7000.
52	18.	5.	0.	246.	14000.	379.	0.	0.	1000.
53	0.	0.	0.	360.	338.	360.	0.	5000.	1500.
54	0.	0.	0.	3700.	1000.	0.	263.	0.	0.
55	0.	50.	0.	1800.	800.	5.	61.	0.	0.
56	0.	0.	0.	1000.	2000.	0.	0.	5000.	0.
57	18.	0.	0.	1000.	158.	0.	0.	800.	25.
58	0.	0.	0.	0.	0.	124.	0.	3.	94.
59	0.	500.	0.	141.	3500.	0.	0.	2000.	60.
60	700.	265.	0.	167.	1500.	0.	0.	17000.	1000.
61	0.	1500.	114.	41.	1000.	0.	0.	439.	0.
62	7000.	130.	0.	130.	9000.	3000.	0.	1700.	1200.
63	0.	0.	0.	0.	12.	3000.	0.	3.	4972.
64	9277.	10000.	27.	13000.	22000.	4800.	1397.	9000.	15500.
65	0.	0.	0.	0.	0.	0.	0.	0.	0.
66	0.	0.	0.	0.	0.	0.	0.	0.	0.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	0.	0.	0.	0.	0.	0.	0.	0.	250.
69	0.	0.	0.	0.	0.	0.	0.	0.	0.
70	0.	0.	0.	0.	0.	0.	0.	0.	0.
71	0.	0.	0.	0.	0.	0.	0.	0.	0.
72	0.	0.	0.	0.	0.	0.	0.	0.	0.
73	0.	0.	0.	0.	0.	0.	0.	0.	0.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	0.	0.	0.	0.	0.	0.	0.	0.	0.
76	0.	0.	0.	0.	0.	0.	0.	0.	0.
77	0.	0.	0.	0.	0.	0.	0.	0.	0.
78	0.	0.	0.	0.	0.	0.	0.	0.	13795.
79	0.	0.	0.	0.	0.	0.	0.	0.	0.

COL. ROW	28	29	30	31	32	33	34	35	36
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	250.	0.	0.	0.	0.	0.	0.	0.	0.
8	0.	0.	0.	334313.	0.	0.	0.	0.	0.
9	0.	0.	0.	11180.	0.	0.	0.	0.	55942.
10	0.	0.	0.	0.	0.	0.	0.	0.	12412.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	0.	0.	0.	0.	0.	0.	0.	0.	0.
13	0.	100.	0.	0.	0.	55.	0.	0.	0.
14	0.	49000.	2500.	100.	1000.	50.	0.	0.	1000.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	0.	0.	0.	2000.	55.	68.	0.	1000.
17	750.	0.	5.	0.	25000.	55.	5.	0.	0.
18	7.	0.	0.	7.	10000.	7.	10000.	0.	7.
19	0.	3125.	5.	1450.	2000.	0.	800.	32.	1250.
20	32.	6875.	1500.	1500.	7000.	0.	1300.	2618.	1500.
21	0.	0.	0.	212.	13.	0.	5.	0.	11.
22	32.	0.	0.	0.	500.	0.	1000.	5000.	0.
23	32.	0.	0.	0.	1000.	0.	26.	5.	6.
24	32.	2000.	18.	1450.	9000.	0.	800.	0.	1000.
25	32.	56.	0.	0.	1000.	0.	800.	0.	202.
26	32.	68.	0.	0.	7000.	0.	6600.	0.	68.
27	316500.	211287.	25000.	57000.	15500.	0.	0.	0.	9000.
28	3406441.	15000.	3000.	15000.	72788.	0.	0.	0.	0.
29	10000.	5779196.	8000.	11000.	1000.	0.	15.	0.	1000.
30	26500.	5622.	1702477.	5979.	1000.	0.	0.	0.	5000.
31	35500.	39000.	7500.	16013897.	1000.	0.	0.	0.	22000.
32	64000.	3000.	55.	1000.	6062644.	3946.	9000.	25.	36000.
33	0.	0.	0.	0.	5000.	862180.	1900.	0.	700.
34	0.	100.	0.	0.	7659.	1900.	2959548.	0.	244.
35	0.	0.	0.	0.	883.	36.	0.	2087515.	5500.
36	1000.	100.	10000.	8500.	12500.	1000.	0.	4000.	6945233.
37	0.	100.	55.	324.	2000.	0.	0.	0.	1000.
38	32.	100.	55.	0.	10000.	0.	900.	0.	307.
39	0.	0.	1.	0.	0.	0.	0.	0.	0.
40	1000.	242.	1000.	2517.	3000.	0.	377.	900.	5000.
41	0.	100.	125.	281.	16500.	125.	94.	900.	4500.
42	25.	5878.	0.	161.	6000.	0.	800.	125.	2600.
43	0.	0.	0.	0.	50.	0.	0.	0.	0.
44	0.	0.	249.	0.	93.	0.	204.	464.	269.
45	0.	0.	0.	0.	80.	0.	0.	0.	500.
46	0.	0.	0.	0.	141.	0.	65.	0.	182.
47	0.	0.	0.	0.	1000.	0.	0.	86.	5300.
48	0.	5600.	30.	0.	5500.	0.	33.	0.	5000.
49	0.	150.	0.	500.	3000.	116.	94.	190.	1000.
50	0.	19.	0.	19.	1000.	19.	19.	19.	1800.
51	0.	83.	0.	0.	0.	0.	0.	0.	0.
52	0.	1500.	0.	0.	153.	0.	0.	429.	492.
53	0.	0.	0.	0.	360.	0.	0.	0.	0.
54	67.	1000.	0.	0.	3000.	0.	0.	830.	26.
55	0.	0.	0.	0.	3500.	0.	1700.	4000.	1000.
56	1000.	0.	0.	4433.	4500.	0.	0.	0.	212.
57	0.	2.	0.	0.	2000.	0.	1900.	5.	1800.
58	0.	122.	0.	0.	1000.	0.	56.	95.	0.
59	0.	208.	0.	241.	110.	55.	110.	0.	1500.
60	0.	0.	0.	0.	16883.	0.	239.	0.	1000.
61	0.	0.	0.	0.	2000.	0.	90.	0.	0.
62	0.	20000.	0.	205.	7000.	0.	5500.	900.	1000.
63	9900.	100.	0.	0.	1500.	0.	3000.	25.	24.
64	27.	1800.	2.	6315.	28500.	27.	8000.	970.	14609.
65	0.	0.	0.	0.	0.	0.	0.	0.	0.
66	0.	0.	0.	0.	0.	0.	0.	0.	0.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	0.	0.	0.	8618.	0.	0.	0.	0.	0.
69	0.	0.	0.	0.	0.	0.	0.	0.	0.
70	0.	0.	0.	0.	0.	0.	0.	0.	0.
71	0.	0.	0.	0.	0.	0.	0.	0.	0.
72	0.	0.	0.	0.	0.	0.	0.	0.	0.
73	0.	0.	0.	0.	0.	0.	0.	0.	0.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	0.	0.	0.	0.	0.	0.	0.	0.	0.
76	0.	0.	0.	0.	0.	0.	0.	0.	0.
77	0.	0.	0.	0.	0.	0.	0.	0.	0.
78	0.	0.	0.	0.	0.	0.	0.	0.	0.
79	0.	0.	0.	0.	0.	0.	0.	0.	7000.

COL. / ROW	37	38	39	40	41	42	43	44	45
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	1875.	1875.	0.	0.	0.	0.	0.	0.	0.
6	0.	171.	0.	0.	0.	0.	0.	0.	0.
7	0.	0.	0.	0.	0.	0.	0.	0.	0.
8	0.	0.	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	0.	0.	0.	0.	0.	0.	0.
10	0.	0.	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	0.	0.	0.	0.	0.	0.	0.	0.	0.
13	1100.	1500.	0.	1000.	2500.	1250.	0.	6500.	1000.
14	0.	0.	0.	0.	0.	214.	0.	150.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	0.	0.	1200.	0.	0.	0.	0.	0.
17	90.	0.	0.	0.	402.	1700.	0.	0.	0.
18	0.	7.	0.	0.	0.	7.	0.	0.	0.
19	0.	73.	0.	2500.	499.	118.	0.	0.	0.
20	3000.	0.	0.	7600.	267.	7000.	0.	5.	281.
21	0.	0.	0.	249.	0.	2955.	0.	500.	0.
22	1500.	0.	39.	8000.	1000.	8000.	0.	100.	500.
23	55.	55.	0.	13000.	500.	6545.	0.	25.	0.
24	78.	0.	0.	1000.	500.	17000.	0.	0.	0.
25	0.	0.	0.	0.	2500.	1250.	0.	0.	0.
26	68.	0.	0.	259.	2500.	12000.	0.	0.	0.
27	11500.	17500.	39.	148.	0.	60.	0.	10.	303.
28	0.	315.	0.	500.	0.	1500.	0.	0.	0.
29	0.	63.	0.	0.	500.	1000.	0.	0.	0.
30	0.	9.	0.	317.	0.	430.	0.	0.	0.
31	25.	0.	25.	33.	0.	25.	0.	0.	0.
32	96.	2000.	39.	1000.	3300.	6600.	0.	0.	500.
33	0.	0.	0.	0.	0.	0.	0.	0.	0.
34	0.	0.	0.	0.	1000.	484.	0.	0.	0.
35	0.	0.	0.	1.	132.	0.	0.	0.	0.
36	3000.	4000.	0.	5600.	1000.	1000.	0.	304.	1000.
37	18152768.	94500.	0.	92500.	72500.	445762.	150.	5000.	3659.
38	44000.	8336500.	2000.	40034.	9500.	72445.	1400.	450.	0.
39	0.	0.	2005733.	3266.	13000.	4707.	0.	0.	112.
40	48000.	3000.	10000.	6763570.	28500.	57000.	2500.	6000.	20000.
41	11000.	9000.	8500.	38000.	2859741.	41000.	7000.	9500.	1000.
42	34000.	15500.	1450.	50000.	37000.	4978194.	125.	6500.	11000.
43	2400.	50.	0.	31000.	0.	7000.	1647910.	1000.	2000.
44	27000.	330.	104.	14000.	4000.	6660.	30000.	2047177.	51500.
45	14500.	79.	0.	24000.	1000.	19000.	61000.	29000.	2523434.
46	2400.	1000.	0.	8000.	1000.	6678.	65.	1000.	17000.
47	16000.	7000.	0.	6100.	15000.	18000.	1450.	5000.	10000.
48	10000.	2000.	8.	29000.	1000.	15387.	2500.	7000.	5600.
49	11000.	4500.	0.	27000.	2500.	41000.	24000.	3000.	21000.
50	2400.	800.	19.	7600.	2500.	8000.	19.	1000.	1000.
51	1100.	0.	0.	2500.	2300.	1000.	0.	0.	1000.
52	1500.	2000.	0.	53000.	6500.	8500.	207.	1000.	2000.
53	3000.	25000.	0.	13000.	9000.	2700.	19000.	2000.	1500.
54	2400.	4000.	5200.	54000.	8000.	13000.	96.	4500.	500.
55	6000.	16000.	39.	9000.	9000.	11000.	50.	50.	1900.
56	208.	9400.	0.	1000.	1000.	2250.	208.	208.	2000.
57	0.	2000.	0.	8000.	2500.	2350.	0.	0.	500.
58	0.	49976.	0.	157.	1000.	2000.	2100.	1000.	0.
59	36000.	14200.	38.	32000.	98000.	31500.	53500.	25000.	44000.
60	1100.	210.	7000.	12000.	4778.	5000.	20000.	5252.	2000.
61	16000.	5835.	0.	57000.	1000.	5000.	20000.	5000.	15000.
62	0.	2000.	0.	7000.	5600.	18000.	130.	198.	1200.
63	78.	9.	0.	436.	1500.	372.	0.	0.	0.
64	2.	7000.	0.	2300.	17000.	16500.	0.	1000.	104.
65	0.	0.	0.	0.	0.	0.	0.	0.	0.
66	0.	0.	0.	0.	0.	0.	0.	0.	0.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	10340.	0.	0.	0.	0.	0.	0.	0.	0.
69	0.	0.	0.	0.	0.	0.	0.	0.	0.
70	0.	0.	0.	0.	0.	0.	0.	0.	0.
71	0.	0.	0.	0.	0.	0.	0.	0.	0.
72	0.	0.	0.	0.	0.	0.	0.	0.	0.
73	0.	0.	0.	0.	0.	0.	0.	0.	0.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	0.	0.	0.	0.	0.	0.	0.	0.	0.
76	0.	0.	0.	0.	0.	0.	0.	0.	0.
77	0.	0.	0.	0.	0.	0.	0.	0.	0.
78	0.	0.	0.	0.	0.	0.	0.	0.	0.
79	0.	0.	0.	0.	0.	0.	0.	0.	0.

COL. ROW	46	47	48	49	50	51	52	53	54
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	0.	0.	0.	0.	0.	0.	0.	0.	0.
8	0.	0.	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	0.	0.	0.	0.	0.	0.	0.
10	0.	0.	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	0.	0.	0.	0.	0.	0.	0.	0.	0.
13	0.	1000.	2000.	478.	67.	2500.	0.	6700.	23.
14	100.	0.	1000.	0.	0.	0.	195.	0.	23.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	0.	57.	360.	2.	0.	0.	0.	0.
17	0.	0.	0.	0.	0.	0.	0.	0.	0.
18	0.	0.	0.	0.	7.	0.	0.	0.	0.
19	0.	0.	0.	0.	2.	0.	118.	0.	0.
20	0.	5.	1500.	220.	0.	0.	1500.	0.	16.
21	0.	0.	0.	0.	0.	0.	418.	0.	0.
22	0.	116.	118.	1300.	0.	0.	2500.	0.	600.
23	500.	5.	55.	500.	1.	500.	1500.	25.	1000.
24	7.	27.	1000.	380.	0.	1000.	171.	0.	0.
25	0.	0.	56.	0.	0.	0.	0.	0.	56.
26	0.	68.	2000.	0.	54.	7000.	163.	0.	0.
27	0.	1.	1400.	1500.	1.	44.	60.	1400.	3.
28	0.	392.	0.	0.	0.	0.	0.	0.	0.
29	0.	27.	57.	500.	67.	0.	1000.	0.	3.
30	0.	0.	0.	0.	0.	0.	0.	0.	0.
31	0.	25.	0.	25.	0.	0.	25.	0.	0.
32	0.	13000.	11000.	468.	1000.	1000.	178.	1000.	23.
33	0.	0.	0.	0.	0.	0.	0.	0.	0.
34	0.	244.	0.	0.	2000.	0.	0.	0.	0.
35	0.	6.	36.	125.	31.	0.	2500.	0.	2100.
36	0.	2000.	1000.	6000.	3000.	1000.	0.	500.	0.
37	1200.	32000.	4500.	25000.	9000.	7500.	266.	1000.	1100.
38	0.	36000.	10000.	16500.	9500.	0.	500.	46000.	1500.
39	1.	1000.	21.	17.	0.	0.	21.	0.	1.
40	5000.	10000.	8000.	47000.	3000.	47.	46000.	3000.	47500.
41	5387.	69500.	6500.	3000.	5000.	41.	1000.	1700.	19500.
42	12000.	61100.	12000.	23000.	15000.	1500.	10000.	8945.	7000.
43	7000.	6600.	1000.	37500.	8000.	0.	1000.	73000.	0.
44	5000.	5600.	7000.	8000.	1000.	38.	1000.	1000.	23.
45	52500.	6950.	19000.	27500.	5000.	26.	1100.	5700.	1100.
46	777466.	5287.	12000.	28500.	600.	38.	2500.	2000.	0.
47	5000.	2746435.	29000.	12000.	5000.	38.	1000.	7000.	1000.
48	4613.	28000.	1961634.	25159.	10000.	12000.	6341.	5821.	23.
49	22000.	20000.	32000.	2868055.	12000.	1000.	23000.	3000.	7000.
50	1800.	17000.	1800.	8500.	1326507.	800.	19.	800.	19.
51	1000.	83.	2000.	122.	1000.	1553671.	171.	3000.	23.
52	2000.	7000.	12000.	32500.	1000.	26.	1660906.	1000.	35500.
53	2000.	14000.	12000.	59500.	384.	5952.	6500.	3983555.	19500.
54	0.	7000.	3000.	0.	134.	1.	143000.	3000.	2902910.
55	1800.	1400.	600.	1500.	50.	0.	95.	45000.	2100.
56	206.	208.	23000.	7000.	1000.	20000.	6500.	51500.	1000.
57	473.	319.	6500.	500.	126.	38.	0.	37000.	64.
58	0.	299.	1000.	3000.	3000.	1000.	0.	11333.	1000.
59	10000.	172500.	6900.	23500.	6400.	118.	28000.	8000.	7500.
60	2000.	20000.	8822.	62500.	5000.	15000.	14500.	5500.	500.
61	4305.	1000.	8000.	19000.	5000.	0.	1000.	26000.	6995.
62	213.	1400.	1400.	7000.	1000.	6626.	1000.	61500.	2200.
63	0.	30.	4678.	500.	67.	38.	0.	3000.	5000.
64	29136.	7000.	1000.	5.	7.	7.	1000.	75.	1500.
65	0.	0.	0.	0.	0.	0.	0.	0.	0.
66	0.	0.	0.	0.	0.	0.	0.	0.	0.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	0.	0.	0.	0.	0.	0.	0.	0.	0.
69	0.	0.	0.	0.	0.	0.	0.	0.	0.
70	0.	0.	0.	0.	0.	0.	0.	0.	0.
71	0.	0.	0.	0.	0.	0.	0.	0.	0.
72	0.	0.	0.	0.	0.	0.	0.	0.	0.
73	0.	0.	0.	0.	0.	0.	0.	0.	0.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	0.	0.	0.	0.	0.	0.	0.	0.	0.
76	0.	0.	0.	0.	0.	0.	0.	0.	0.
77	0.	0.	0.	0.	0.	0.	0.	0.	0.
78	0.	0.	0.	0.	0.	0.	0.	0.	0.
79	0.	0.	0.	0.	0.	0.	0.	0.	0.

COL. ROW	55	56	57	58	59	60	61	62	63
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	0.	0.	0.	0.	0.	0.	0.	0.	0.
8	0.	0.	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	0.	0.	0.	0.	0.	0.	0.
10	0.	0.	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	0.	0.	0.	0.	0.	0.	0.	0.	0.
13	19.	145500.	22000.	0.	6200.	457642.	0.	0.	0.
14	0.	0.	0.	0.	26.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	0.	0.	0.	0.	0.	0.	0.	0.
17	15.	0.	0.	0.	1200.	0.	0.	0.	0.
18	0.	0.	0.	0.	0.	0.	7.	0.	0.
19	0.	88.	0.	0.	2700.	0.	44.	0.	0.
20	1200.	0.	0.	38.	1500.	1500.	2500.	0.	0.
21	0.	0.	0.	38.	0.	0.	0.	0.	0.
22	19.	0.	0.	0.	1200.	0.	1000.	0.	0.
23	7.	55.	51.	0.	55.	5400.	1500.	0.	0.
24	0.	0.	18250.	0.	0.	0.	0.	0.	0.
25	31.	0.	0.	0.	0.	0.	0.	0.	0.
26	0.	1000.	36.	0.	0.	150.	44.	0.	0.
27	19.	0.	0.	0.	29.	0.	0.	0.	0.
28	0.	0.	0.	0.	0.	0.	0.	0.	0.
29	0.	0.	0.	0.	0.	0.	0.	0.	0.
30	19.	0.	0.	0.	0.	0.	0.	0.	0.
31	0.	0.	0.	0.	0.	0.	0.	0.	0.
32	19.	86.	36.	1100.	31000.	0.	44.	0.	0.
33	0.	0.	0.	0.	0.	0.	0.	0.	0.
34	0.	0.	0.	0.	0.	0.	0.	0.	0.
35	19.	0.	0.	0.	1000.	0.	0.	0.	0.
36	2000.	36.	1000.	0.	2000.	3000.	0.	0.	0.
37	11000.	0.	3646.	0.	1200.	0.	10000.	0.	0.
38	11000.	203.	1000.	8600.	1000.	18951.	0.	0.	0.
39	19.	0.	0.	0.	21.	0.	0.	0.	0.
40	5260.	1000.	1000.	38.	16000.	7000.	0.	0.	0.
41	3600.	89.	36.	1000.	9000.	1000.	0.	0.	0.
42	5000.	89.	1000.	1200.	5500.	11000.	0.	0.	0.
43	0.	0.	0.	1800.	25000.	17915.	0.	0.	0.
44	700.	0.	0.	1000.	23967.	2000.	0.	0.	0.
45	700.	1000.	2000.	2200.	9000.	2000.	0.	0.	0.
46	19.	65.	0.	0.	5000.	5000.	0.	0.	0.
47	700.	2000.	36.	1100.	1200.	9000.	0.	0.	0.
48	19.	2000.	1000.	33.	5700.	5500.	0.	0.	0.
49	3126.	6280.	1000.	38.	24000.	15000.	0.	0.	0.
50	19.	0.	19.	9100.	12500.	900.	0.	0.	0.
51	0.	7000.	12500.	900.	0.	5000.	0.	0.	0.
52	2000.	89.	36.	50.	2000.	7000.	0.	0.	0.
53	42500.	44333.	77000.	24000.	21000.	7000.	0.	0.	0.
54	700.	2000.	2000.	4000.	13.	28000.	0.	0.	0.
55	1883359.	6400.	10000.	31000.	4800.	1000.	0.	0.	0.
56	5000.	4720699.	87000.	7000.	6600.	35425.	0.	0.	0.
57	8000.	54500.	2076976.	10000.	1100.	2000.	0.	0.	0.
58	22000.	2000.	2000.	1176380.	26000.	150.	0.	0.	0.
59	19.	1000.	0.	4600C.	21385584.	40500.	0.	0.	0.
60	19.	51500.	0.	1100.	9000.	9580256.	0.	0.	0.
61	19.	1000.	0.	38.	17500.	1900.	3322025.	0.	0.
62	5274.	22000.	10000.	5100.	2400.	36424.	0.	2558465.	0.
63	0.	14000.	500.	500.	29.	500.	0.	0.	1293509.
64	5100.	7500.	1800.	27.	27.	14500.	0.	0.	0.
65	0.	0.	0.	0.	0.	0.	0.	0.	0.
66	0.	0.	0.	0.	0.	0.	0.	0.	0.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	0.	0.	0.	0.	0.	0.	0.	0.	0.
69	0.	0.	0.	0.	0.	0.	0.	0.	0.
70	0.	0.	0.	0.	0.	0.	0.	0.	0.
71	0.	0.	0.	0.	0.	0.	0.	0.	0.
72	0.	0.	0.	0.	0.	0.	0.	0.	0.
73	0.	0.	0.	0.	0.	0.	0.	0.	0.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	0.	0.	0.	0.	0.	0.	0.	0.	0.
76	0.	0.	0.	0.	0.	0.	0.	0.	0.
77	0.	0.	0.	0.	0.	0.	0.	0.	0.
78	0.	0.	0.	0.	0.	0.	0.	0.	0.
79	0.	0.	0.	0.	0.	0.	0.	0.	0.

COL. ROW	64	65	66	67	68	69	70	71	72
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	0.	0.	0.	0.	0.	0.	0.	0.	0.
8	0.	0.	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	0.	0.	0.	0.	0.	0.	0.
10	0.	0.	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	0.	0.	0.	0.	0.	0.	0.	0.	0.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	0.	0.	0.	0.	0.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	0.	0.	0.	0.	0.	0.	0.	0.
17	0.	0.	0.	0.	0.	0.	0.	0.	0.
18	0.	0.	0.	0.	0.	0.	0.	0.	0.
19	0.	0.	0.	0.	0.	0.	0.	0.	0.
20	0.	0.	0.	0.	0.	0.	0.	0.	0.
21	0.	0.	0.	0.	0.	0.	0.	0.	0.
22	0.	0.	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.	0.	0.
24	0.	0.	0.	0.	0.	0.	0.	0.	0.
25	0.	0.	0.	0.	0.	0.	0.	0.	0.
26	0.	0.	0.	0.	0.	0.	0.	0.	0.
27	0.	0.	0.	0.	0.	0.	0.	0.	0.
28	0.	0.	0.	0.	0.	0.	0.	0.	0.
29	0.	0.	0.	0.	0.	0.	0.	0.	0.
30	0.	0.	0.	0.	0.	0.	0.	0.	0.
31	0.	0.	0.	0.	0.	0.	0.	0.	0.
32	0.	0.	0.	0.	0.	0.	0.	0.	0.
33	0.	0.	0.	0.	0.	0.	0.	0.	0.
34	0.	0.	0.	0.	0.	0.	0.	0.	0.
35	0.	0.	0.	0.	0.	0.	0.	0.	0.
36	0.	0.	0.	0.	0.	0.	0.	0.	0.
37	0.	0.	0.	0.	0.	0.	0.	0.	0.
38	0.	0.	0.	0.	0.	0.	0.	0.	0.
39	0.	0.	0.	0.	0.	0.	0.	0.	0.
40	0.	0.	0.	0.	0.	0.	0.	0.	0.
41	0.	0.	0.	0.	0.	0.	0.	0.	0.
42	0.	0.	0.	0.	0.	0.	0.	0.	0.
43	0.	0.	0.	0.	0.	0.	0.	0.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	0.	0.	0.	0.	0.	0.	0.	0.	0.
46	0.	0.	0.	0.	0.	0.	0.	0.	0.
47	0.	0.	0.	0.	0.	0.	0.	0.	0.
48	0.	0.	0.	0.	0.	0.	0.	0.	0.
49	0.	0.	0.	0.	0.	0.	0.	0.	0.
50	0.	0.	0.	0.	0.	0.	0.	0.	0.
51	0.	0.	0.	0.	0.	0.	0.	0.	0.
52	0.	0.	0.	0.	0.	0.	0.	0.	0.
53	0.	0.	0.	0.	0.	0.	0.	0.	0.
54	0.	0.	0.	0.	0.	0.	0.	0.	0.
55	0.	0.	0.	0.	0.	0.	0.	0.	0.
56	0.	0.	0.	0.	0.	0.	0.	0.	0.
57	0.	0.	0.	0.	0.	0.	0.	0.	0.
58	0.	0.	0.	0.	0.	0.	0.	0.	0.
59	0.	0.	0.	0.	0.	0.	0.	0.	0.
60	0.	0.	0.	0.	0.	0.	0.	0.	0.
61	0.	0.	0.	0.	0.	0.	0.	0.	0.
62	0.	0.	0.	0.	0.	0.	0.	0.	0.
63	0.	0.	0.	0.	0.	0.	0.	0.	0.
64	4223820.	0.	0.	0.	0.	0.	0.	0.	0.
65	0.	29502960.	0.	0.	0.	0.	0.	0.	0.
66	0.	0.	8886570.	0.	0.	0.	0.	0.	0.
67	0.	0.	0.	16602.	0.	0.	0.	0.	0.
68	0.	0.	0.	0.	17152496.	0.	0.	0.	0.
69	0.	0.	0.	0.	0.	90945104.	0.	0.	0.
70	0.	0.	0.	0.	0.	0.	25632464.	0.	0.
71	0.	0.	0.	0.	0.	0.	0.	55274304.	0.
72	0.	0.	0.	0.	0.	0.	0.	0.	10848678.
73	0.	0.	0.	0.	0.	0.	0.	0.	0.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	0.	0.	0.	0.	0.	0.	0.	0.	0.
76	0.	0.	0.	0.	0.	0.	0.	0.	0.
77	0.	0.	0.	0.	0.	0.	0.	0.	0.
78	0.	0.	0.	0.	0.	0.	0.	0.	0.
79	0.	0.	0.	0.	0.	0.	0.	0.	0.

COL. ROW	73	74	75	76	77	78	79
1	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.
7	0.	0.	0.	0.	0.	0.	0.
8	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	0.	0.	0.	0.	0.
10	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.
12	0.	0.	0.	0.	0.	0.	0.
13	0.	0.	0.	0.	0.	0.	0.
14	0.	0.	0.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.
16	0.	0.	0.	0.	0.	0.	0.
17	0.	0.	0.	0.	0.	0.	0.
18	0.	0.	0.	0.	0.	0.	0.
19	0.	0.	0.	0.	0.	0.	0.
20	0.	0.	0.	0.	0.	0.	0.
21	0.	0.	0.	0.	0.	0.	0.
22	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.
24	0.	0.	0.	0.	0.	0.	0.
25	0.	0.	0.	0.	0.	0.	0.
26	0.	0.	0.	0.	0.	0.	0.
27	0.	0.	0.	0.	0.	0.	0.
28	0.	0.	0.	0.	0.	0.	0.
29	0.	0.	0.	0.	0.	0.	0.
30	0.	0.	0.	0.	0.	0.	0.
31	0.	0.	0.	0.	0.	0.	0.
32	0.	0.	0.	0.	0.	0.	0.
33	0.	0.	0.	0.	0.	0.	0.
34	0.	0.	0.	0.	0.	0.	0.
35	0.	0.	0.	0.	0.	0.	0.
36	0.	0.	0.	0.	0.	0.	0.
37	0.	0.	0.	0.	0.	0.	0.
38	0.	0.	0.	0.	0.	0.	0.
39	0.	0.	0.	0.	0.	0.	0.
40	0.	0.	0.	0.	0.	0.	0.
41	0.	0.	0.	0.	0.	0.	0.
42	0.	0.	0.	0.	0.	0.	0.
43	0.	0.	0.	0.	0.	0.	0.
44	0.	0.	0.	0.	0.	0.	0.
45	0.	0.	0.	0.	0.	0.	0.
46	0.	0.	0.	0.	0.	0.	0.
47	0.	0.	0.	0.	0.	0.	0.
48	0.	0.	0.	0.	0.	0.	0.
49	0.	0.	0.	0.	0.	0.	0.
50	0.	0.	0.	0.	0.	0.	0.
51	0.	0.	0.	0.	0.	0.	0.
52	0.	0.	0.	0.	0.	0.	0.
53	0.	0.	0.	0.	0.	0.	0.
54	0.	0.	0.	0.	0.	0.	0.
55	0.	0.	0.	0.	0.	0.	0.
56	0.	0.	0.	0.	0.	0.	0.
57	0.	0.	0.	0.	0.	0.	0.
58	0.	0.	0.	0.	0.	0.	0.
59	0.	0.	0.	0.	0.	0.	0.
60	0.	0.	0.	0.	0.	0.	0.
61	0.	0.	0.	0.	0.	0.	0.
62	0.	0.	0.	0.	0.	0.	0.
63	0.	0.	0.	0.	0.	0.	0.
64	0.	0.	0.	0.	0.	0.	0.
65	0.	0.	0.	0.	0.	0.	0.
66	0.	0.	0.	0.	0.	0.	0.
67	0.	0.	0.	0.	0.	0.	0.
68	0.	0.	0.	0.	0.	0.	0.
69	0.	0.	0.	0.	0.	0.	0.
70	0.	0.	0.	0.	0.	0.	0.
71	0.	0.	0.	0.	0.	0.	0.
72	0.	0.	0.	0.	0.	0.	0.
73	15879226.	0.	0.	0.	0.	0.	0.
74	0.	534270.	0.	0.	0.	0.	0.
75	0.	0.	7822002.	0.	0.	0.	0.
76	0.	0.	0.	5370848.	0.	0.	0.
77	0.	0.	0.	0.	22103552.	0.	0.
78	0.	0.	0.	0.	0.	3144265.	0.
79	0.	0.	0.	0.	0.	0.	741900.

TABLE F-4

"INDUSTRY-PRODUCT MIX" MATRIX, UNITED STATES ECONOMY, 1958

U = [u_kl] k, l = 1, ..., 79.

COL. ROW	1	2	3	4	5	6	7	8	9
1	0.99973	0.0	0.06618	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	1.00000	0.13393	0.34995	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.79988	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.65005	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.97531	0.01270	0.0	0.0	0.00028
6	0.0	0.0	0.0	0.0	0.02469	0.98602	0.0	0.0	0.00066
7	0.0	0.0	0.0	0.0	0.0	0.0	0.99984	0.0	0.00065
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.00000	0.0
9	0.0	0.0	0.0	0.0	0.0	0.00007	0.00016	0.0	0.92908
10	0.0	0.0	0.0	0.0	0.0	0.00121	0.0	0.0	0.00047
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.00027	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
17	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
18	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
19	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
21	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
27	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
28	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
29	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
30	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
31	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
32	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
33	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
34	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
35	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
36	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.06883
37	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
38	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
40	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
41	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
42	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
43	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
46	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
47	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
48	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
49	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
53	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
55	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
56	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
57	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
58	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
59	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
60	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
63	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
64	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
65	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
66	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
69	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
70	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
71	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
72	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
73	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
76	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
77	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
78	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
79	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

COL. / ROW	10	11	12	13	14	15	16	17	18
1	0.0	0.0	0.0	0.0	0.02131	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.00140	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.00109	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.00093	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.00371	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
10	0.55388	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	1.00000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.0	0.0	1.00000	0.0	0.0	0.0	0.0	0.0	0.0
13	0.0	0.0	0.0	0.81921	0.0	0.0	0.0	0.0	0.0
14	0.0	0.0	0.0	0.0	0.97558	0.00002	0.0	0.00001	0.0
15	0.0	0.0	0.0	0.0	0.00001	0.99995	0.0	0.0	0.0
16	0.0	0.0	0.0	0.0	0.00005	0.0	0.98498	0.03219	0.00177
17	0.0	0.0	0.0	0.0	0.00000	0.0	0.00982	0.92465	0.00005
18	0.0	0.0	0.0	0.0	0.00002	0.0	0.00164	0.00347	0.00564
19	0.0	0.0	0.0	0.0	0.0	0.0	0.00278	0.00599	0.00085
20	0.0	0.0	0.0	0.0	0.00000	0.0	0.00001	0.00001	0.0
21	0.0	0.0	0.0	0.0	0.00000	0.0	0.0	0.0	0.00000
22	0.0	0.0	0.0	0.00003	0.0	0.0	0.00002	0.00314	0.00000
23	0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.0	0.0
24	0.0	0.0	0.0	0.0	0.00000	0.0	0.00002	0.00574	0.00000
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00001	0.0
26	0.0	0.0	0.0	0.0	0.0	0.00003	0.00002	0.0	0.00000
27	0.04039	0.0	0.0	0.0	0.00053	0.0	0.0	0.0	0.0
28	0.0	0.0	0.0	0.0	0.00005	0.0	0.00015	0.00499	0.0
29	0.0	0.0	0.0	0.00000	0.00086	0.0	0.00000	0.0	0.00000
30	0.0	0.0	0.0	0.0	0.00000	0.0	0.0	0.00045	0.0
31	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
32	0.0	0.0	0.0	0.00000	0.00005	0.0	0.00030	0.01597	0.00043
33	0.0	0.0	0.0	0.0	0.00000	0.0	0.0	0.0	0.00001
34	0.0	0.0	0.0	0.00004	0.0	0.0	0.00003	0.00045	0.00078
35	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
36	0.0	0.0	0.0	0.0	0.00001	0.00003	0.00015	0.00095	0.00000
37	0.0	0.0	0.0	0.00392	0.0	0.0	0.0	0.0	0.0
38	0.0	0.0	0.0	0.00107	0.0	0.0	0.0	0.00001	0.0
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00001	0.0
40	0.0	0.0	0.0	0.00320	0.0	0.0	0.00035	0.0	0.0
41	0.0	0.0	0.0	0.00071	0.0	0.0	0.0	0.0	0.0
42	0.0	0.0	0.0	0.00071	0.0	0.0	0.00001	0.00173	0.00002
43	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
44	0.0	0.0	0.0	0.00178	0.0	0.0	0.0	0.00045	0.0
45	0.0	0.0	0.0	0.00071	0.0	0.0	0.0	0.0	0.0
46	0.0	0.0	0.0	0.00002	0.0	0.0	0.0	0.00001	0.0
47	0.0	0.0	0.0	0.00071	0.0	0.0	0.0	0.00001	0.0
48	0.0	0.0	0.0	0.00192	0.00000	0.0	0.0	0.0	0.00000
49	0.0	0.0	0.0	0.00071	0.00000	0.0	0.0	0.0	0.0
50	0.0	0.0	0.0	0.00001	0.00000	0.0	0.0	0.0	0.0
51	0.0	0.0	0.0	0.00142	0.0	0.0	0.0	0.0	0.0
52	0.0	0.0	0.0	0.00655	0.00000	0.0	0.0	0.0	0.0
53	0.0	0.0	0.0	0.00356	0.0	0.0	0.0	0.0	0.0
54	0.0	0.0	0.0	0.00036	0.0	0.0	0.0	0.0	0.0
55	0.0	0.0	0.0	0.00036	0.0	0.0	0.0	0.0	0.0
56	0.0	0.0	0.0	0.00641	0.0	0.0	0.0	0.0	0.0
57	0.0	0.0	0.0	0.00007	0.0	0.0	0.0	0.0	0.0
58	0.0	0.0	0.0	0.00036	0.0	0.0	0.0	0.0	0.0
59	0.0	0.0	0.0	0.00712	0.0	0.0	0.0	0.00045	0.0
60	0.0	0.0	0.0	0.12426	0.0	0.0	0.0	0.00585	0.0
61	0.0	0.0	0.0	0.00597	0.0	0.0	0.0	0.0	0.00000
62	0.0	0.0	0.0	0.00605	0.00000	0.0	0.00002	0.00001	0.00014
63	0.0	0.0	0.0	0.00107	0.0	0.0	0.00003	0.0	0.00000
64	0.0	0.0	0.0	0.00178	0.00000	0.0	0.00000	0.00001	0.00028
65	0.0	0.0	0.0	0.0	0.00009	0.0	0.0	0.0	0.0
66	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
69	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
70	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
71	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
72	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
73	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
76	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
77	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
78	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
79	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

COL. ROW	19	20	21	22	23	24	25	26	27
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.02630	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00075
6	0.0	0.00003	0.0	0.0	0.0	0.0	0.0	0.0	0.00002
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.00002	0.0	0.0	0.0	0.0	0.0	0.0	0.00001
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.01335
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.00000	0.0	0.0	0.0	0.0	0.0	0.00037	0.00090	0.0
14	0.0	0.00001	0.0	0.00000	0.00007	0.00015	0.00007	0.0	0.00522
15	0.0	0.0	0.0	0.0	0.0	0.00001	0.00001	0.0	0.0
16	0.15658	0.00000	0.0	0.0	0.0	0.00011	0.0	0.00013	0.00001
17	0.00517	0.00000	0.0	0.00189	0.00216	0.00078	0.0	0.00013	0.00013
18	0.00858	0.00000	0.00002	0.00000	0.0	0.00000	0.00000	0.00000	0.0
19	0.80240	0.00006	0.0	0.00079	0.00090	0.00269	0.00009	0.00000	0.00006
20	0.00001	0.96325	0.07265	0.00850	0.00504	0.00062	0.00085	0.00000	0.00013
21	0.0	0.00266	0.91283	0.00000	0.00006	0.00000	0.00057	0.00001	0.0
22	0.00316	0.00240	0.00654	0.96142	0.02698	0.00002	0.0	0.0	0.0
23	0.00280	0.00039	0.00121	0.00772	0.89665	0.00017	0.0	0.00282	0.00000
24	0.03788	0.00012	0.0	0.0	0.00072	0.97442	0.00753	0.01107	0.10250
25	0.0	0.00000	0.00242	0.0	0.0	0.02549	0.97635	0.00326	0.0
26	0.00001	0.00001	0.0	0.00002	0.00058	0.00655	0.00440	0.97332	0.0
27	0.00001	0.00019	0.0	0.0	0.0	0.00016	0.00002	0.00000	0.96444
28	0.0	0.00025	0.0	0.0	0.0	0.00004	0.00034	0.0	0.01946
29	0.00046	0.00001	0.0	0.0	0.00002	0.00000	0.00006	0.00016	0.01281
30	0.0	0.0	0.0	0.0	0.0	0.00011	0.0	0.0	0.00301
31	0.0	0.00006	0.0	0.0	0.0	0.00011	0.0	0.00013	0.05173
32	0.00083	0.00019	0.00030	0.00208	0.00108	0.00129	0.00242	0.00016	0.00176
33	0.00001	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00001
34	0.00278	0.00006	0.0	0.00031	0.00000	0.00011	0.00014	0.00004	0.0
35	0.00001	0.00000	0.0	0.00001	0.00000	0.0	0.00195	0.0	0.00000
36	0.0	0.00039	0.0	0.00008	0.00023	0.00426	0.0	0.0	0.00046
37	0.00037	0.0	0.00036	0.00005	0.00108	0.00011	0.0	0.0	0.01107
38	0.0	0.00001	0.0	0.0	0.00017	0.00011	0.00014	0.0	0.00621
39	0.0	0.0	0.00005	0.00001	0.0	0.00006	0.00403	0.00000	0.0
40	0.00001	0.00039	0.00056	0.00331	0.01367	0.00004	0.0	0.0	0.00018
41	0.00001	0.00117	0.0	0.00167	0.00381	0.00022	0.0	[illegible]	0.00065
42	0.00065	0.00039	0.00030	0.00783	0.00216	0.00073	0.00008	0.00163	0.00020
43	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
44	0.0	0.00003	0.0	0.0	0.00072	0.0	0.0	0.0	0.00002
45	0.0	0.00000	0.0	0.0	0.00036	0.0	0.0	0.00006	0.00000
46	0.0	0.00005	0.0	0.00063	0.00005	0.0	0.0	0.0	0.00001
47	0.0	0.00000	0.0	0.00003	0.00004	0.00002	0.00007	0.00002	0.00055
48	0.0	0.00006	0.00242	0.00176	0.00003	0.00011	0.00002	0.00001	0.00000
49	0.00037	0.00004	0.0	0.00009	0.00144	0.00002	0.0	0.0	0.00018
50	0.0	[illegible]	0.0	0.0	0.00001	0.0	0.0	0.0	0.00000
51	0.00001	0.0	0.0	0.0	0.00218	0.00019	0.0	0.00078	0.00065
52	0.00001	0.00000	0.0	0.00008	0.01007	0.00004	0.0	0.0	0.00005
53	0.0	0.0	0.0	0.00011	0.00024	0.00004	0.0	0.00002	0.00014
54	0.0	0.0	0.0	0.00117	0.00072	0.0	0.00027	0.0	0.0
55	0.0	0.00001	0.0	0.00057	0.00058	0.00000	0.00002	0.0	0.0
56	0.0	0.0	0.0	0.00031	0.00144	0.0	0.0	0.00002	0.0
57	0.00001	0.0	0.0	0.00031	0.00011	0.0	0.0	0.00013	0.00000
58	0.0	0.0	0.0	0.0	0.0	0.00001	0.0	0.00000	0.00001
59	0.0	0.00006	0.0	0.00004	0.00252	0.0	0.0	0.00033	0.00001
60	0.00032	0.00003	0.0	0.00005	0.00108	0.0	0.0	0.00277	0.00000
61	0.0	0.00019	0.00028	0.00001	0.00072	0.0	0.0	0.00007	0.0
62	0.00425	0.00002	0.0	0.00004	0.10648	0.00034	0.0	0.00028	0.0
63	0.0	0.0	0.0	0.0	0.00001	0.00034	0.0	0.00000	0.00011
64	0.00433	0.00130	0.00007	0.00459	0.01583	0.00054	0.00040	0.00147	0.00143
65	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
66	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	[illegible]
69	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
70	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
71	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
72	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
73	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
76	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
77	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
78	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00128
79	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

COL. ROW	28	29	30	31	32	33	34	35	36
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.00006	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.0	0.0	0.02028	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.00068	0.0	0.0	0.0	0.0	0.00792
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00173
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.0	0.00002	0.0	0.0	0.0	0.00006	0.0	0.0	0.0
14	0.0	0.00797	0.00142	0.00001	0.00016	0.00006	0.0	0.0	0.00014
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.0	0.0	0.0	0.00031	0.00006	0.00002	0.0	0.00014
17	0.00019	0.0	0.00000	0.0	0.00393	0.00006	0.00000	0.0	0.0
18	0.00000	0.0	0.0	0.00000	0.00157	0.00001	0.00332	0.0	0.00000
19	0.0	0.00051	0.00000	0.00009	0.00031	0.0	0.00027	0.00002	0.00017
20	0.00001	0.00112	0.00085	0.00009	0.00110	0.0	0.00043	0.00124	0.00021
21	0.0	0.0	0.0	0.00001	0.00000	0.0	0.00000	0.0	0.00000
22	0.00001	0.0	0.0	0.0	0.00008	0.0	0.00033	0.00237	0.0
23	0.00001	0.0	0.0	0.0	0.00016	0.0	0.00001	0.00000	0.00000
24	0.00001	0.00033	0.00001	0.00009	0.00141	0.0	0.00027	0.0	0.00014
25	0.00001	0.00001	0.0	0.0	0.00016	0.0	0.00027	0.0	0.00003
26	0.00001	0.00001	0.0	0.0	0.00110	0.0	0.00215	0.0	0.00001
27	0.08172	0.03435	0.01419	0.00346	0.00244	0.0	0.0	0.0	0.00126
28	0.87949	0.00244	0.00170	0.00051	0.01144	0.0	0.0	0.0	0.0
29	0.00253	0.93948	0.00454	0.00067	0.00016	0.0	0.00000	0.0	0.00014
30	0.00684	0.00091	0.96645	0.00036	0.00016	0.0	0.0	0.0	0.00070
31	0.00917	0.00634	0.00426	0.97135	0.00016	0.0	0.0	0.0	0.00307
32	0.01652	0.00049	0.00003	0.00006	0.95259	0.00454	0.00299	0.00001	0.00503
33	0.0	0.0	0.0	0.0	0.00079	0.99144	0.00063	0.0	0.00016
34	0.0	0.00002	0.0	0.0	0.00120	0.00218	0.98155	0.0	0.00003
35	0.0	0.0	0.0	0.0	0.00014	0.00004	0.0	0.98975	0.00077
36	0.00026	0.00002	0.00568	0.00052	0.00196	0.00115	0.0	0.00190	0.97066
37	0.0	0.00002	0.00003	0.00002	0.00031	0.0	0.0	0.0	0.00014
38	0.00001	0.00002	0.00003	0.0	0.00157	0.0	0.00030	0.0	0.00004
39	0.0	0.0	0.00000	0.0	0.0	0.0	0.0	0.0	0.0
40	0.00025	0.00004	0.00057	0.00015	0.00047	0.0	0.00013	0.00043	0.00070
41	0.0	0.00002	0.00007	0.00002	0.00259	0.00014	0.00003	0.00043	0.00063
42	0.00001	0.00196	0.0	0.00001	0.00094	0.0	0.00027	[illegible]	0.00036
43	0.0	0.0	0.0	0.0	0.00001	0.0	0.0	0.0	0.0
44	0.0	0.0	0.00014	0.0	0.00001	0.0	0.00017	0.00022	0.00004
45	0.0	0.0	0.0	0.0	0.00001	0.0	0.0	0.0	0.00007
46	0.0	0.0	0.0	0.0	0.00002	0.0	0.00002	0.0	0.00073
47	0.0	0.0	0.0	0.0	0.00016	0.0	0.0	0.00004	0.00074
48	0.0	0.00091	0.00002	0.0	0.00086	0.0	0.00001	0.0	0.00070
49	0.0	0.00002	0.0	0.00003	0.00047	0.00013	0.00003	0.00009	0.00014
50	0.0	0.00000	0.0	0.00000	0.00016	0.00002	0.00001	0.00001	0.00025
51	0.0	0.00001	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.00024	0.0	0.0	0.00002	0.0	0.0	0.00020	0.00007
53	0.0	0.0	0.0	0.0	0.00006	0.0	0.0	0.0	0.0
54	0.00002	0.00016	0.0	0.0	0.00047	0.0	0.0	[illegible]	0.00000
55	0.0	0.0	0.0	0.0	0.00055	0.0	0.00056	0.00190	0.00014
56	0.00026	0.0	0.0	0.00027	0.00071	0.0	0.0	0.0	0.00033
57	0.0	0.00000	0.0	0.0	0.00031	0.0	0.00063	0.00000	0.00025
58	0.0	0.00002	0.0	0.0	0.00016	0.0	0.00002	0.00035	0.0
59	0.0	0.00003	0.0	0.00001	0.00002	0.00006	0.00004	0.0	0.00021
60	0.0	0.0	0.0	0.0	0.00265	0.0	0.00009	0.0	0.00014
61	0.0	0.0	0.0	0.0	0.00031	0.0	0.00003	0.0	0.0
62	0.0	0.00325	0.0	0.00001	0.00110	0.0	0.00182	0.00043	0.00014
63	0.00256	0.00002	0.0	0.0	0.00024	0.0	0.00190	0.00001	0.00000
64	0.00001	0.00029	0.00000	0.00038	0.00448	0.00003	0.00265	0.00046	0.00234
65	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
66	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.0	0.0	0.0	0.00052	0.0	0.0	0.0	0.0	0.0
69	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
70	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
71	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
72	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
73	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
76	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
77	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
78	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
79	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00098

COL. / ROW	37	38	39	40	41	42	43	44	45
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.00010	0.00022	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.00002	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.00006	0.00017	0.0	0.00013	0.00076	0.00021	0.0	0.00299	[illegible]
14	0.0	0.0	0.0	0.0	0.0	0.00004	0.0	0.00007	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.0	0.0	0.00016	0.0	0.0	0.0	0.0	0.0
17	0.00000	0.0	0.0	0.0	0.00012	0.00029	0.0	0.0	0.0
18	0.0	0.00000	0.0	0.0	0.0	0.00000	0.0	0.0	0.0
19	0.0	0.00001	0.0	0.00034	0.00015	0.00002	0.0	0.0	0.0
20	0.00016	0.0	0.0	0.00102	0.00008	0.00118	0.0	0.00000	0.00012
21	0.0	0.0	0.0	0.00003	0.0	0.00050	0.0	0.00023	0.0
22	0.00008	0.0	0.00002	0.00108	0.00031	0.00135	0.0	0.00005	0.00018
23	0.00000	0.00001	0.0	0.00175	0.00015	0.00111	0.0	0.00001	0.0
24	0.00000	0.0	0.0	0.00013	0.00015	0.00288	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.00076	0.00021	0.0	0.0	0.0
26	0.00000	0.0	0.0	0.00003	0.00076	0.00203	0.0	0.0	0.0
27	0.00062	0.00203	0.00002	0.00002	0.0	0.00001	0.0	[illegible]	0.00011
28	0.0	0.00004	0.0	0.00007	0.0	0.00025	0.0	0.0	0.0
29	0.0	0.00001	0.0	0.0	0.00015	0.00017	0.0	0.0	0.0
30	0.0	0.00000	0.0	0.00004	0.0	0.00007	0.0	0.0	0.0
31	0.00000	0.0	0.00001	0.00000	0.0	0.00000	0.0	0.0	0.0
32	0.00001	0.00023	0.00002	0.00013	0.00101	0.00112	0.0	0.0	0.00018
33	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
34	0.0	0.0	0.0	0.0	0.00031	0.00008	0.0	0.0	0.0
35	0.0	0.0	0.0	0.00000	0.00004	0.0	0.0	0.0	0.0
36	0.00016	0.00045	0.0	0.00075	0.00031	0.00017	0.0	0.00014	[illegible]
37	0.99253	0.01094	0.0	0.01245	0.02216	0.07544	0.00008	[illegible]	0.00133
38	0.00238	0.96516	0.00098	0.00575	0.00250	0.01226	0.00074	0.00021	0.0
39	0.0	0.0	0.98309	0.00044	0.00397	0.00080	0.0	0.0	[illegible]
40	0.00263	0.00035	0.00490	0.91057	0.00871	0.00965	0.00132	0.00276	0.00729
41	0.00060	0.00093	0.00417	0.00512	0.98649	0.00654	0.00370	0.00391	0.00036
42	0.00184	0.00179	0.00071	0.07673	0.01131	0.84246	0.00007	0.00288	[illegible]
43	0.00013	0.00001	0.0	0.00417	0.0	0.00118	0.87034	0.00046	[illegible]
44	0.00146	0.00074	0.00005	0.00188	0.00122	0.00113	0.01584	0.94191	0.01876
45	0.00076	0.00021	0.0	0.00323	0.00031	0.00322	0.03222	0.01324	0.02005
46	0.00013	0.00012	0.0	0.00108	0.00031	0.00113	0.00073	0.00346	0.00620
47	0.00087	0.00001	0.0	0.00082	0.00459	0.00305	0.00077	0.00230	0.00365
48	0.00054	0.00023	0.00000	0.00350	0.00031	0.00260	0.00132	0.00322	0.01294
49	[illegible]	0.00052	0.0	0.00363	0.00076	0.00654	0.01268	0.00133	0.01766
50	0.00013	0.00009	0.00001	0.00102	0.00076	0.00135	0.00001	[illegible]	[illegible]
51	0.00036	0.0	0.0	0.00034	0.00070	0.00017	0.0	0.0	0.00036
52	[illegible]	0.00023	0.0	0.00714	0.00199	0.00144	0.00011	0.00045	0.00073
53	0.00016	0.00289	0.0	0.00175	0.00275	0.00046	0.01003	0.00092	0.00055
54	0.00013	0.00046	0.00255	0.00727	0.00245	0.00220	[illegible]	[illegible]	0.00018
55	0.00032	0.00185	0.00002	0.00121	0.00275	0.00186	0.00093	0.00092	0.00069
56	0.00001	0.00097	0.0	0.00013	0.00031	0.00638	0.00011	0.00012	0.00073
57	0.0	0.00023	0.0	0.00108	0.00076	0.00040	0.0	0.0	0.00018
58	0.0	0.00579	0.0	0.00002	0.00031	0.00034	0.00111	0.00545	0.0
59	0.00195	0.00164	0.00002	0.00431	0.02996	0.00533	0.02826	[illegible]	0.01604
60	0.00076	0.00002	0.00343	0.00162	0.00146	0.00085	0.01056	0.00242	0.00073
61	[illegible]	0.00068	0.0	0.00767	0.00031	0.00085	0.01056	0.00230	0.00547
62	0.0	0.00023	0.0	0.00094	0.00171	0.00305	0.00007	0.00029	0.00044
63	0.00000	0.00000	0.0	0.00006	0.00046	0.00006	0.0	0.0	0.0
64	0.00000	0.00001	0.0	0.00031	0.00520	0.00279	0.0	0.00045	0.00004
65	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
66	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.00056	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
69	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
70	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
71	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
72	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
73	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
76	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
77	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
78	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
79	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

COL. / ROW	46	47	48	49	50	51	52	53	54
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.0	0.00030	0.00090	0.00014	0.00005	0.00153	0.0	0.00152	0.00001
14	0.00010	0.0	0.00045	0.0	0.0	0.0	0.00010	0.0	0.00001
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.0	0.00003	0.00011	0.00000	0.0	0.0	0.0	0.0
17	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
18	0.0	0.0	0.0	0.0	0.00000	0.0	0.0	0.0	0.0
19	0.0	0.0	0.0	0.0	0.00000	0.0	0.00006	0.0	0.0
20	0.0	0.00000	0.00068	0.00007	0.0	0.0	0.00076	0.0	0.00001
21	0.0	0.0	0.0	0.0	0.0	0.0	0.00021	0.0	0.0
22	0.0	0.00004	0.00005	0.00039	0.0	0.0	0.00127	0.0	0.00020
23	0.00052	0.00000	0.00002	0.00015	0.00000	0.00031	0.00076	0.00001	0.00033
24	0.00001	0.00001	0.00045	0.00011	0.0	0.00061	0.00009	0.0	0.0
25	0.0	0.0	0.00003	0.0	0.0	0.0	0.0	0.0	0.00002
26	0.0	0.00002	0.00090	0.0	0.00004	0.00427	0.00008	0.0	0.0
27	0.0	0.00000	0.00063	0.00045	0.00000	0.00003	0.00003	0.00032	0.00000
28	0.0	0.00012	0.0	0.0	0.0	0.0	0.0	0.0	0.0
29	0.0	0.00001	0.00003	0.00015	0.00005	0.0	0.00051	0.0	0.00000
30	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
31	0.0	0.00001	0.0	0.00001	0.0	0.0	0.00001	0.0	0.0
32	0.0	0.00395	0.00496	0.00014	0.00069	0.00061	0.00009	0.00023	0.00001
33	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
34	0.0	0.00007	0.0	0.0	0.00135	0.0	0.0	0.0	0.0
35	0.0	0.00000	0.00002	0.00004	0.00002	0.0	0.00127	0.0	0.00068
36	0.0	0.00061	0.00045	0.00179	0.00208	0.00061	0.0	0.00011	0.0
37	0.00126	0.00971	0.00203	0.00744	0.00625	0.00458	0.00014	0.00023	0.00036
38	0.0	0.01093	0.00451	0.00451	0.00660	0.0	0.00025	0.01044	0.00049
39	0.00002	0.00030	0.00001	0.00001	0.0	0.0	0.00001	0.0	0.00000
40	0.00525	0.00304	0.00361	0.01400	0.00208	0.00003	0.02339	0.00069	0.01545
41	0.00565	0.02109	0.00293	0.00089	0.00347	0.00003	0.00051	0.00032	0.00634
42	0.01267	0.01855	0.00541	0.00685	0.01041	0.00092	0.00508	0.00020	0.00028
43	0.00735	0.00200	0.00045	0.01117	0.00555	0.0	0.00051	0.01656	0.0
44	0.00525	0.00177	0.00316	0.00238	0.00069	0.00002	0.00051	0.00023	0.00001
45	0.05511	0.00211	0.00857	0.00819	0.00347	0.00002	0.00056	0.00129	0.00036
46	0.81695	0.00160	0.00541	0.00845	0.00042	0.00002	0.00127	0.00045	0.0
47	0.00525	0.83361	0.01308	0.00357	0.00347	0.00002	0.00051	0.00159	0.00033
48	0.00484	0.00350	0.88496	0.00749	0.00694	0.00732	0.00322	0.00132	0.00001
49	0.02309	0.00607	0.01444	0.85404	0.00833	0.00061	0.01169	0.00068	0.00228
50	0.00185	0.00516	0.00081	0.00253	0.92120	0.00049	0.00001	0.00019	0.00001
51	0.00105	0.00003	0.00090	0.00004	0.00069	0.94820	0.00009	0.00068	0.00001
52	0.00210	0.00212	0.00541	0.00568	0.00069	0.00002	0.84449	0.00023	0.01154
53	0.00210	0.00425	0.00541	0.01772	0.00027	0.00363	0.00330	0.90370	0.00634
54	0.0	0.00212	0.00135	0.0	0.00005	0.00000	0.07271	0.00068	0.94391
55	0.00184	0.00042	0.00027	0.00045	0.00003	0.0	0.00005	0.01021	0.00068
56	0.00022	0.00006	0.01038	0.00208	0.00069	0.01221	0.00330	0.01169	0.00032
57	0.00050	0.00010	0.00293	0.00015	0.00009	0.00002	0.0	0.00439	0.00032
58	0.0	0.00009	0.00045	0.00089	0.00208	0.00061	0.0	0.00257	0.00032
59	[illegible]	0.05236	0.00311	0.00700	0.00444	0.00007	0.01424	0.00181	0.00244
60	0.00210	0.00607	0.00398	0.01861	0.00347	0.00915	0.00737	0.00125	0.00016
61	0.00452	0.00030	0.00361	0.00566	0.00347	0.0	0.00051	0.00590	0.00227
62	0.00022	0.00042	0.00063	0.00208	0.00069	0.00404	0.00051	0.01355	0.00072
63	0.0	0.00001	0.00211	0.00015	0.00005	0.00002	0.0	0.00068	0.00163
64	0.03057	0.00212	0.00045	0.00000	0.00000	0.00000	0.00051	0.00000	0.00049
65	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
66	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
69	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
70	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
71	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
72	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
73	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
76	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
77	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
78	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
79	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

COL. ROW	55	56	57	58	59	60	61	62	63
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.00001	0.02857	0.00943	0.0	0.00029	0.04433	0.0	0.03428	0.00146
14	0.0	0.0	0.0	0.0	0.00000	0.0	0.0	0.0	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.0
17	0.00001	0.0	0.0	0.0	0.00006	0.0	0.0	0.00000	0.00001
18	0.0	0.0	0.0	0.0	0.0	0.0	0.00000	0.00214	0.0
19	0.0	0.00002	0.0	0.0	0.00012	0.0	0.00001	0.00042	0.0
20	0.00059	0.0	0.0	0.00003	0.00007	0.00015	0.00073	0.00050	0.0
21	0.0	0.0	0.0	0.00003	0.0	0.0	0.0	0.0	0.0
22	0.00001	0.0	0.0	0.0	0.00006	0.0	0.00029	0.00074	0.0
23	0.00000	0.00001	0.00002	0.0	0.00000	0.00052	0.00044	0.00435	0.00000
24	0.0	0.0	0.00783	0.0	0.0	0.0	0.0	0.00767	0.00073
25	0.00002	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	0.0	0.00020	0.00002	0.0	0.0	0.00001	0.00001	0.00000	0.00001
27	0.00001	0.0	0.0	0.0	0.00000	0.0	0.0	0.00059	0.00087
28	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00001	0.0
29	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00214	0.00000
30	0.00001	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
31	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00001	0.0
32	0.00001	0.00002	0.00002	0.00082	0.00143	0.0	0.00001	0.00067	0.0
33	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
34	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00033	0.0
35	0.00001	0.0	0.0	0.0	0.00005	0.0	0.0	0.00000	0.00073
36	0.02059	0.00001	0.00043	0.0	0.00009	0.00029	0.0	0.00001	0.01320
37	0.03545	0.0	0.00156	0.0	0.00006	0.0	0.00290	0.0	[illegible]
38	0.00545	0.00004	0.00043	0.00645	0.00005	0.00184	0.00002	0.00064	0.00013
39	0.00001	0.0	0.0	0.0	0.00000	0.0	0.00000	0.0	0.0
40	0.00261	0.00020	0.00043	0.00003	0.00074	0.00768	0.00465	0.00064	0.0
41	0.00178	0.00002	0.00002	0.00075	0.00042	0.00010	0.00020	0.00001	0.00017
42	0.00248	0.00002	0.00043	0.00050	0.00025	0.00107	0.00154	0.00602	0.00270
43	0.0	0.0	0.0	0.00135	0.00115	0.00174	0.00424	0.00012	0.0
44	0.00035	0.0	0.0	0.00075	0.00111	0.00019	0.00552	0.00040	0.0
45	0.00035	0.00020	0.00086	0.00165	0.00042	0.00019	0.00058	0.00074	0.00029
46	0.00001	0.00001	0.0	0.0	0.00023	0.00048	0.00087	0.00001	0.0
47	0.00035	0.00039	0.00002	0.00082	0.00006	0.00087	0.00029	0.00074	0.00066
48	0.00001	0.00037	0.00043	0.00077	0.00026	0.00053	0.00020	0.00204	0.00145
49	0.00155	0.00123	0.00043	0.00003	0.00111	0.00145	0.00058	0.00224	0.0
50	0.00001	0.0	0.00001	0.00682	0.00058	0.00009	0.00001	0.00275	0.00073
51	0.0	0.00137	0.00536	0.00067	0.0	0.00048	0.0	0.00853	0.00058
52	0.00795	0.00002	0.00002	0.00004	0.00009	0.00078	0.00094	0.00177	0.0
53	0.02105	0.00370	0.03302	0.01800	0.00057	0.00068	0.00058	0.01605	0.00510
54	0.00035	0.00039	0.00086	0.00300	0.00000	0.00271	0.0	0.00100	0.0
55	0.93304	0.00126	0.00429	0.02325	0.00022	0.00010	0.00029	0.00033	0.00146
56	0.00243	0.92681	0.03731	0.00525	0.00030	0.00343	0.0	0.00736	0.00513
57	0.00396	0.01070	0.89067	0.00750	0.00005	0.00019	0.0	0.00214	0.00016
58	[illegible]	0.00039	0.00086	0.88225	0.00120	0.00001	0.0	0.00100	[illegible]
59	0.00001	0.00020	0.0	0.03450	0.98724	0.00392	0.00610	0.01706	0.0
60	0.00001	0.00111	0.0	0.00082	0.00042	0.92808	0.00494	0.01722	0.00007
61	0.00001	0.00020	0.0	0.00003	0.00081	0.00018	0.96459	0.00167	0.00059
62	0.00261	0.00432	0.00429	0.00382	0.00011	0.00353	0.00001	0.85562	0.01528
63	0.0	0.00275	0.00021	0.00037	0.00000	0.00005	0.0	0.00535	0.94163
64	0.00253	0.00147	0.00077	0.00002	0.00000	0.00140	0.00029	0.00190	0.00146
65	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
66	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
69	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
70	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
71	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
72	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
73	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
76	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
77	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
78	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
79	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

COL. ROW	64	65	66	67	68	69	70	71	72
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.01373	0.0
2	0.0	0.00088	0.0	0.0	0.0	0.0	0.0	0.02091	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00002	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00109	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00008	0.0
6	0.0	0.0	0.0	0.0	0.00014	0.00000	0.0	0.00008	0.0
7	0.0	0.0	0.0	0.0	0.00012	0.00004	0.0	0.00017	0.0
8	0.0	0.0	0.0	0.0	0.00387	0.00000	0.0	0.00155	0.0
9	0.00001	0.0	0.0	0.0	0.00001	0.00004	0.0	0.00007	0.0
10	0.0	0.0	0.0	0.0	0.00000	0.00000	0.0	0.00002	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.00011	0.0	0.0	0.0	0.00001	0.00010	0.0	0.00006	0.0
14	0.00020	0.0	0.0	0.0	0.00002	0.00562	0.0	0.00101	0.0
15	0.00013	0.0	0.0	0.0	0.00001	0.00003	0.0	0.00003	0.0
16	0.00000	0.0	0.0	0.0	0.00009	0.00010	0.0	0.00026	0.0
17	0.00017	0.0	0.0	0.0	0.00000	0.00008	0.0	0.00009	0.0
18	0.00100	0.0	0.0	0.0	0.0	0.00056	0.0	0.00042	0.0
19	0.00111	0.0	0.0	0.0	0.0	0.00018	0.0	0.00005	0.0
20	0.00156	0.0	0.0	0.0	0.00013	0.00058	0.0	0.00029	0.0
21	0.00001	0.0	0.0	0.0	0.0	0.00003	0.0	0.00003	0.0
22	0.00033	0.0	0.0	0.0	0.00000	0.00014	0.0	0.00005	0.0
23	0.00111	0.0	0.0	0.0	0.0	0.00014	0.0	0.00002	0.0
24	0.00156	0.0	0.0	0.0	0.00036	0.00029	0.0	0.00017	0.0
25	0.00067	0.0	0.0	0.0	0.00002	0.00006	0.0	0.00015	0.0
26	0.00356	0.0	0.0	0.0	0.00000	0.00065	0.0	0.00001	0.0
27	0.00151	0.0	0.0	0.0	0.00047	0.00107	0.0	0.00054	0.0
28	0.0	0.0	0.0	0.0	0.00010	0.00004	0.0	0.00017	0.0
29	0.00222	0.0	0.0	0.0	0.00001	0.00127	0.0	0.00043	0.0
30	0.0	0.0	0.0	0.0	0.00000	0.00024	0.0	0.00017	0.0
31	0.0	0.0	0.0	0.0	0.00016	0.00042	0.0	0.00190	0.0
32	0.01567	0.0	0.0	0.0	0.00003	0.00086	0.0	0.00017	0.0
33	0.00010	0.0	0.0	0.0	0.00000	0.00002	0.0	0.00001	0.0
34	0.00156	0.0	0.0	0.0	0.00000	0.00016	0.0	0.00006	0.0
35	0.00020	0.0	0.0	0.0	0.0	0.00005	0.0	0.00009	0.0
36	0.00544	0.0	0.0	0.0	0.00004	0.00083	0.0	0.00032	0.0
37	0.00044	0.0	0.0	0.0	0.00180	0.00008	0.0	0.00033	0.0
38	0.00002	0.0	0.0	0.0	0.00042	0.00017	0.0	0.00018	0.0
39	0.0	0.0	0.0	0.0	0.0	0.00004	0.0	0.00014	0.0
40	0.00044	0.0	0.0	0.0	0.0	0.00104	0.0	0.00015	0.0
41	0.00567	0.0	0.0	0.0	0.0	0.00025	0.0	0.00008	0.0
42	0.00245	0.0	0.0	0.0	0.0	0.00057	0.0	0.00011	0.0
43	0.00001	0.0	0.0	0.0	0.00009	0.00012	0.0	0.00007	0.0
44	0.00038	0.0	0.0	0.0	0.00001	0.00019	0.0	0.00010	0.0
45	0.00001	0.0	0.0	0.0	0.0	0.00030	0.0	0.00012	0.0
46	0.00001	0.0	0.0	0.0	0.0	0.00014	0.0	0.00007	0.0
47	0.00044	0.0	0.0	0.0	0.0	0.00016	0.0	0.00055	0.0
48	0.00044	0.0	0.0	0.0	0.0	0.00030	0.0	0.00018	0.0
49	0.00044	0.0	0.0	0.0	0.0	0.00031	0.0	0.00014	0.0
50	0.00020	0.0	0.0	0.0	0.0	0.00010	0.0	0.00014	0.0
51	0.00311	0.0	0.0	0.0	0.0	0.00039	0.0	0.00013	0.0
52	0.00001	0.0	0.0	0.0	0.0	0.00032	0.0	0.00001	0.0
53	0.00044	0.0	0.0	0.0	0.00001	0.00025	0.0	0.00011	0.0
54	0.00245	0.0	0.0	0.0	0.0	0.00026	0.0	0.00017	0.0
55	0.00040	0.0	0.0	0.0	0.0	0.00015	0.0	0.00005	0.0
56	0.00300	0.0	0.0	0.0	0.00001	0.00066	0.0	0.00012	0.0
57	0.00044	0.0	0.0	0.0	0.0	0.00010	0.0	0.00022	0.0
58	0.00020	0.0	0.0	0.0	0.0	0.00029	0.0	0.00034	0.0
59	0.00133	0.0	0.0	0.0	0.00004	0.00047	0.0	0.00010	0.0
60	0.00076	0.0	0.0	0.0	0.00000	0.00072	0.0	0.00001	0.0
61	0.00245	0.0	0.0	0.0	0.00000	0.00013	0.0	0.00011	0.0
62	0.00137	0.0	0.0	0.0	0.0	0.00045	0.0	0.00013	0.0
63	0.00001	0.0	0.0	0.0	0.00001	0.00038	0.0	0.00008	0.0
64	0.03888	0.0	0.0	0.0	0.00001	0.00071	0.0	0.00073	0.0
65	0.0	0.97614	0.0	0.0	0.00020	0.00053	0.0	0.00565	0.00017
66	0.0	0.0	1.00000	0.0	0.0	0.0	0.0	0.00072	0.0
67	0.0	0.0	0.0	1.00000	0.0	0.0	0.0	0.00033	0.0
68	0.0	0.0	0.0	0.0	0.94878	0.0	0.0	[illegible]	0.0
69	0.0	0.0	0.0	0.0	0.0	0.96758	0.0	0.01454	0.0
70	0.0	0.0	0.0	0.0	0.0	0.0	0.99755	0.01241	0.0
71	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.80247	0.0
72	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.33492	0.49093
73	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00018	0.0
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00113	0.0
76	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00158	0.0
77	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00041	0.0
78	0.0	0.00041	0.0	0.0	0.01696	0.00601	0.0	0.00044	0.0
79	0.0	0.02257	0.0	0.0	0.12604	0.00328	0.00245	0.00668	0.0

COL. ROW	73	74	75	76	77	78	79
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.0	0.0	0.0	0.0	0.0
8	0.0	0.00009	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.0	0.0	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.0	0.19828	0.0	0.0	0.0	0.0	0.0
14	0.0	0.00075	0.0	0.0	0.0	0.0	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.00019	0.0	0.0	0.0	0.0	0.0
17	0.0	0.00019	0.0	0.0	0.0	0.0	0.0
18	0.0	0.00019	0.0	0.0	0.0	0.0	0.0
19	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0	0.0	0.0	0.0
21	0.0	0.0	0.0	0.0	0.0	0.0	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.0	0.0	0.0	0.0	0.0	0.0	0.0
25	0.0	0.0	0.0	0.0	0.0	0.0	0.0
26	0.22569	0.0	0.0	0.0	0.0	0.0	0.0
27	0.0	0.01573	0.0	0.0	0.0	0.0	0.0
28	0.0	0.00150	0.0	0.0	0.0	0.0	0.0
29	0.0	0.00328	0.0	0.0	0.0	0.0	0.0
30	0.0	0.0	0.0	0.0	0.0	0.0	0.0
31	0.0	0.00352	0.0	0.0	0.0	0.0	0.0
32	0.0	0.00225	0.0	0.0	0.0	0.0	0.0
33	0.0	0.0	0.0	0.0	0.0	0.0	0.0
34	0.0	0.0	0.0	0.0	0.0	0.0	0.0
35	0.0	0.00019	0.0	0.0	0.0	0.0	0.0
36	0.0	0.0	0.0	0.0	0.0	0.0	0.0
37	0.0	0.00054	0.0	0.0	0.0	0.0	0.0
38	0.0	0.00243	0.0	0.0	0.0	0.0	0.0
39	0.0	0.00019	0.0	0.0	0.0	0.0	0.0
40	0.0	0.00251	0.0	0.0	0.0	0.0	0.0
41	0.0	0.00056	0.0	0.0	0.0	0.0	0.0
42	0.0	0.00337	0.0	0.0	0.0	0.0	0.0
43	0.0	0.00109	0.0	0.0	0.0	0.0	0.0
44	0.0	0.00225	0.0	0.0	0.0	0.0	0.0
45	0.0	0.00071	0.0	0.0	0.0	0.0	0.0
46	0.0	0.00056	0.0	0.0	0.0	0.0	0.0
47	0.0	0.01310	0.0	0.0	0.0	0.0	0.0
48	0.0	0.00567	0.0	0.0	0.0	0.0	0.0
49	0.0	0.00056	0.0	0.0	0.0	0.0	0.0
50	0.0	0.00041	0.0	0.0	0.0	0.0	0.0
51	0.01576	0.00225	0.0	0.0	0.0	0.0	0.0
52	0.0	0.00169	0.0	0.0	0.0	0.0	0.0
53	0.0	0.03165	0.0	0.0	0.0	0.0	0.0
54	0.0	0.02809	0.0	0.0	0.0	0.0	0.0
55	0.0	0.01348	0.0	0.0	0.0	0.0	0.0
56	0.0	0.04786	0.0	0.0	0.0	0.0	0.0
57	0.0	0.02641	0.0	0.0	0.0	0.0	0.0
58	0.0	0.00431	0.0	0.0	0.0	0.0	0.0
59	0.0	0.03446	0.0	0.0	0.0	0.0	0.0
60	0.0	0.23996	0.0	0.0	0.0	0.0	0.0
61	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.0	0.02033	0.0	0.0	0.0	0.0	0.0
63	0.0	0.00585	0.0	0.0	0.0	0.0	0.0
64	0.00730	0.00193	0.0	0.0	0.0	0.0	0.0
65	0.0	0.0	0.0	0.0	0.0	0.0	0.0
66	0.01485	0.0	0.0	0.0	0.0	0.0	0.0
67	0.06357	0.0	0.0	0.0	0.0	0.0	0.0
68	0.0	0.0	0.0	0.0	0.0	0.0	0.0
69	0.0	0.0	0.0	0.0	0.0	0.0	0.0
70	0.0	0.0	0.0	0.0	0.0	0.0	0.0
71	0.0	0.0	0.0	0.0	0.0	0.0	0.0
72	0.00506	0.0	0.0	0.0	0.0	0.0	0.0
73	0.66519	0.0	0.0	0.0	0.0	0.0	0.0
74	0.0	0.11006	0.0	0.0	0.0	0.0	0.0
75	0.0	0.0	0.99732	0.0	0.0	0.0	0.0
76	0.00138	0.0	0.0	1.00000	0.0	0.0	0.0
77	0.0	0.10076	0.0	0.0	1.00000	0.0	0.0
78	0.0	0.0	0.0	0.0	0.0	1.00000	0.0
79	0.0	0.0	0.00268	0.0	0.0	0.0	1.00000

TABLE F-5

"COMMODITY-TO-COMMODITY" FLOWS MATRIX, UNITED STATES ECONOMY, 1958

(thousands of 1958 dollars)

$$X = [x^D_{ij} + x^M_{ij}] \quad i,j = 1, \ldots, 79.$$

COL. / ROW	1	2	3	4	5	6	7	8	9
1	3881857.	1615477.	0.	259811.	0.	0.	0.	0.	0.
2	6174554.	672193.	15334.	37268.	0.	0.	0.	0.	0.
3	0.	0.	19009.	0.	0.	0.	0.	0.	0.
4	489609.	846681.	22170.	24559.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	64570.	0.	0.	0.	0.
6	0.	0.	0.	0.	18179.	231660.	0.	0.	0.
7	4966.	431.	0.	0.	4634.	1069.	470549.	147.	1073.
8	0.	0.	0.	0.	0.	0.	0.	233428.	0.
9	605.	65069.	0.	1.	0.	0.	978.	0.	8591.
10	0.	26231.	78.	0.	0.	0.	63.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	144321.	263780.	271.	2082.	491.	886.	1974.	3879.	1499.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	2753083.	2760.	31002.	11572.	0.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	7010.	0.	0.	128.	1859.	1856.	0.	0.
17	4718.	23712.	14451.	12142.	0.	0.	0.	2136.	42.
18	0.	0.	0.	0.	0.	0.	0.	0.	0.
19	5827.	32799.	0.	0.	0.	0.	0.	0.	0.
20	1714.	1319.	0.	0.	5901.	1083.	18228.	5484.	0.
21	0.	96828.	0.	111.	0.	0.	0.	0.	0.
22	0.	0.	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.	0.	0.
24	0.	0.	10195.	3013.	0.	506.	6315.	3492.	12749.
25	12675.	2536.	11705.	9369.	0.	0.	1477.	5.	2502.
26	4888.	14621.	109.	4.	0.	3810.	731.	2759.	481.
27	29804.	1117184.	217.	488.	10954.	42129.	44047.	47126.	16807.
28	0.	0.	0.	0.	0.	0.	0.	0.	0.
29	24760.	0.	0.	76.	4.	108.	0.	4.	181.
30	0.	0.	2294.	0.	0.	200.	876.	4887.	0.
31	42314.	834029.	23678.	4014.	10420.	8024.	30406.	49896.	45954.
32	18625.	147308.	10711.	2802.	521.	3807.	22838.	30581.	30236.
33	0.	0.	0.	0.	0.	0.	0.	0.	0.
34	819.	3532.	18.	3.	3.	4.	8.	42.	4.
35	1935.	0.	0.	0.	0.	0.	0.	403.	0.
36	989.	23803.	0.	0.	689.	6229.	6332.	3338.	484.
37	0.	0.	0.	0.	20398.	51457.	22868.	2949.	23052.
38	791.	740.	0.	0.	1617.	6486.	16825.	7252.	1764.
39	4817.	12061.	0.	0.	0.	0.	0.	0.	0.
40	0.	0.	0.	0.	1161.	431.	1008.	5902.	15.
41	17786.	0.	0.	0.	257.	596.	15794.	5495.	189.
42	26774.	34381.	879.	13881.	796.	1333.	14159.	45339.	776.
43	0.	0.	163.	0.	389.	527.	0.	14677.	3.
44	4789.	190302.	0.	0.	0.	0.	0.	0.	23.
45	0.	0.	0.	0.	23687.	28040.	97986.	38035.	77025.
46	0.	0.	0.	0.	3.	111.	10105.	0.	21720.
47	0.	0.	0.	0.	47.	727.	9364.	0.	85.
48	0.	0.	0.	0.	0.	7.	0.	8582.	0.
49	0.	0.	0.	0.	182.	2392.	4836.	76519.	5423.
50	1871.	2624.	0.	0.	50.	93.	566.	714.	521.
51	0.	0.	0.	0.	0.	0.	0.	0.	0.
52	0.	0.	0.	0.	0.	0.	0.	0.	0.
53	0.	0.	0.	0.	709.	4861.	5718.	32613.	3039.
54	0.	0.	0.	0.	0.	0.	0.	0.	0.
55	687.	831.	130.	0.	369.	504.	4177.	677.	127.
56	0.	0.	0.	0.	1269.	0.	0.	0.	0.
57	0.	0.	0.	0.	0.	0.	0.	11017.	0.
58	7074.	20177.	0.	0.	233.	164.	266.	1091.	284.
59	23231.	33124.	0.	0.	538.	982.	5652.	3817.	6257.
60	0.	0.	0.	0.	0.	0.	0.	0.	0.
61	0.	2872.	24998.	0.	2462.	0.	11334.	0.	277.
62	0.	0.	0.	0.	193.	422.	140.	553.	319.
63	0.	0.	0.	0.	60.	71.	42.	23.	111.
64	306.	464.	2890.	782.	8.	5.	3979.	410.	535.
65	460812.	245895.	8654.	16804.	24163.	14527.	20502.	79602.	10820.
66	46083.	61713.	4039.	9967.	1549.	1927.	1916.	1427.	2545.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	79093.	150055.	568.	1758.	16627.	29179.	64789.	57570.	38834.
69	857935.	973840.	22857.	6740.	17495.	25361.	95039.	96986.	63453.
70	155942.	226368.	36862.	7756.	5767.	16942.	26985.	101189.	17448.
71	269559.	1621025.	48754.	24585.	80097.	36693.	60384.	1374588.	41558.
72	0.	0.	0.	0.	915.	1072.	646.	430.	1501.
73	24500.	709538.	127223.	7543.	5717.	7789.	13272.	332978.	14170.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	52323.	45582.	0.	0.	0.	0.	933.	13694.	289.
76	0.	0.	0.	0.	22.	26.	10.	11.	29.
77	140370.	10818.	1110.	1378.	824.	1064.	2594.	3960.	1508.
78	3519.	2923.	1139.	1510.	743.	964.	2673.	0.	927.
79	391.	585.	118.	269.	76.	371.	615.	4166.	1519.

COL. ROW	10	11	12	13	14	15	16	17	18
1	0.	0.	0.	0.	15297189.	0.	102117.	57193.	0.
2	0.	236829.	0.	0.	5123291.	1094949.	1196975.	3453.	6482.
3	0.	0.	0.	0.	268527.	0.	0.	0.	133964.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	353.	58.	0.	0.	40553.	1320.	15547.	1318.	603.
8	0.	50.	0.	0.	0.	0.	0.	0.	0.
9	7627.	625278.	131148.	0.	3507.	0.	0.	6.	0.
10	26403.	0.	0.	0.	7028.	0.	694.	0.	287.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	174.	7000.	1000.	6550.	223517.	244.	6311.	370.	5881.
13	0.	5291.	0.	81086.	0.	0.	0.	0.	0.
14	40.	16801.	0.	0.	10771420.	35076.	22264.	16107.	0.
15	0.	0.	0.	0.	0.	1133231.	0.	0.	0.
16	146.	0.	0.	0.	2240.	1113.	3580217.	331338.	3902721.
17	0.	3866.	1005.	77.	497.	0.	112828.	218725.	74958.
18	0.	0.	0.	2208.	38946.	0.	0.	0.	2459673.
19	0.	355.	724.	0.	98273.	0.	4772.	36.	160412.
20	141.	3280061.	417781.	742.	3138.	1261.	1626.	0.	0.
21	0.	0.	0.	4891.	99028.	9146.	0.	0.	0.
22	0.	296438.	0.	0.	0.	0.	0.	0.	0.
23	0.	204643.	16470.	0.	0.	0.	0.	0.	0.
24	2096.	323259.	68095.	4263.	375168.	69003.	14449.	13350.	13305.
25	331.	0.	0.	148900.	892741.	69441.	80046.	14344.	87218.
26	18.	87697.	2727.	7062.	763274.	97206.	51052.	8267.	52133.
27	3199.	366665.	70740.	10085.	172261.	5132.	167146.	4961.	41744.
28	0.	0.	0.	0.	10867.	105718.	919442.	407054.	143354.
29	103.	0.	0.	2813.	158268.	7432.	23119.	831.	1167.
30	0.	196763.	879311.	1813.	5688.	141.	3606.	36.	287.
31	3604.	986432.	374904.	9418.	286190.	2640.	25613.	3953.	5113.
32	1566.	311100.	65922.	106171.	142602.	9757.	35610.	1081.	15421.
33	0.	0.	0.	0.	0.	215.	1734.	284.	34945.
34	1.	316.	52.	0.	221.	4.	54.	10.	79.
35	0.	85252.	81512.	2482.	609421.	0.	23087.	1258.	25.
36	187.	4085008.	548016.	10759.	1668.	26.	417.	0.	10.
37	8296.	2225798.	273666.	42875.	712.	0.	3864.	744.	633.
38	1618.	868834.	281417.	208352.	35954.	6810.	2593.	231.	0.
39	0.	0.	0.	0.	1531303.	7542.	0.	0.	0.
40	92.	5192256.	881419.	0.	0.	0.	0.	0.	0.
41	370.	88205.	20905.	28470.	178284.	342.	1096.	119.	0.
42	335.	869305.	52195.	41524.	86808.	8893.	8329.	747.	16975.
43	157.	2120.	354.	5008.	0.	0.	0.	0.	0.
44	0.	2563.	0.	0.	0.	0.	0.	0.	0.
45	13368.	170611.	21220.	0.	0.	0.	0.	0.	0.
46	2469.	249867.	8256.	0.	0.	0.	0.	0.	0.
47	0.	1007.	181.	46342.	13908.	332.	2107.	389.	6.
48	0.	0.	0.	0.	0.	0.	60632.	908.	0.
49	726.	272914.	18777.	24191.	1650.	200.	1074.	234.	159.
50	49.	2552.	394.	375738.	825.	106.	234.	0.	0.
51	0.	0.	0.	0.	0.	0.	0.	0.	0.
52	0.	196015.	21760.	0.	0.	0.	0.	0.	0.
53	2428.	423820.	78780.	84268.	7976.	42.	584.	120.	0.
54	0.	205565.	59816.	0.	0.	0.	0.	0.	0.
55	0.	794987.	122952.	50783.	22716.	586.	2695.	419.	0.
56	0.	36749.	21674.	131403.	0.	0.	0.	0.	0.
57	0.	1891.	196.	0.	0.	0.	0.	0.	0.
58	49.	14601.	4896.	0.	3487.	32.	91.	0.	33.
59	1164.	1304.	217.	0.	0.	0.	0.	0.	0.
60	0.	0.	0.	589324.	0.	0.	0.	0.	0.
61	184.	2500.	0.	0.	0.	0.	0.	0.	0.
62	69.	191247.	15843.	77694.	12.	0.	36.	15.	16.
63	25.	5.	1.	0.	0.	0.	0.	0.	0.
64	79.	84097.	48370.	6790.	30936.	6534.	10222.	20171.	257434.
65	10448.	1807429.	298023.	47916.	2618747.	71439.	253542.	48952.	117606.
66	252.	108419.	18086.	18994.	156050.	2234.	15900.	6008.	42256.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	18156.	150032.	25028.	14986.	350582.	5530.	115658.	15493.	43575.
69	7757.	4962172.	1378711.	123985.	2364517.	77090.	301504.	95393.	499865.
70	1651.	435017.	49233.	29995.	333242.	11809.	60439.	16396.	92624.
71	3917.	209759.	34992.	16045.	265073.	7281.	43249.	16378.	158441.
72	340.	0.	0.	3897.	37145.	2461.	16723.	2693.	36362.
73	2678.	2474327.	58637.	44640.	952810.	177633.	51954.	9239.	91552.
74	0.	0.	0.	0.	4847.	0.	1962.	0.	0.
75	0.	263865.	22408.	0.	282433.	1791.	5582.	1174.	2344.
76	10.	0.	0.	125.	799.	47.	327.	52.	701.
77	348.	58065.	9686.	4359.	64791.	5887.	11031.	2154.	16363.
78	269.	0.	0.	3099.	27952.	12473.	7500.	2856.	25365.
79	77.	13252.	2211.	572.	29788.	340.	2314.	433.	1215.

COL. ROW	19	20	21	22	23	24	25	26	27
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	12607.
3	815.	797960.	0.	0.	0.	0.	0.	0.	15824.
4	0.	8488.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	52211.
6	0.	0.	0.	0.	0.	0.	0.	0.	68257.
7	0.	1425.	0.	2147.	0.	74656.	1387.	0.	65587.
8	0.	0.	0.	0.	0.	0.	0.	0.	211999.
9	0.	0.	0.	0.	0.	36396.	0.	0.	20545.
10	0.	31.	0.	0.	0.	14751.	0.	0.	223182.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	328.	14106.	39.	1629.	267.	39871.	12236.	40195.	5349.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	144.	0.	0.	29247.	0.	77301.	0.	0.	77124.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	805969.	0.	0.	193323.	996.	50366.	0.	0.	411.
17	186743.	1022.	0.	36981.	23848.	7993.	0.	16375.	197.
18	980.	11255.	0.	1050.	1619.	6968.	3264.	0.	5594.
19	167671.	986.	0.	1188.	96.	0.	0.	0.	31589.
20	0.	2442858.	137776.	376620.	82471.	649026.	0.	385.	31566.
21	0.	0.	17831.	255.	0.	5429.	0.	0.	3201.
22	0.	0.	0.	48594.	4521.	0.	0.	0.	0.
23	0.	0.	0.	0.	30704.	0.	0.	0.	0.
24	7539.	57597.	798.	11736.	3556.	2009723.	1561808.	2104942.	77245.
25	24866.	35819.	0.	61752.	29279.	231647.	113344.	40583.	68318.
26	8180.	38796.	938.	31783.	2766.	76756.	8952.	1913248.	79556.
27	196.	59926.	0.	151.	149.	334343.	6534.	180565.	2435993.
28	0.	58091.	0.	812.	1462.	91391.	9806.	0.	16060.
29	361.	9036.	241.	68.	19.	21293.	5547.	5003.	5908.
30	288.	38935.	314.	68750.	26122.	1478.	0.	0.	11074.
31	1708.	79311.	2714.	6749.	2894.	122299.	24632.	8844.	133937.
32	79039.	52840.	236.	131650.	11351.	133256.	23488.	11946.	65755.
33	0.	0.	0.	3662.	2113.	0.	0.	0.	0.
34	0.	0.	3.	0.	0.	0.	0.	98.	102.
35	0.	9059.	0.	45884.	60103.	0.	0.	0.	14142.
36	0.	30669.	621.	8128.	2117.	10932.	97.	0.	32555.
37	0.	308.	18852.	73764.	145381.	0.	1112.	0.	102327.
38	0.	8042.	0.	27293.	18515.	10362.	0.	13191.	107511.
39	0.	0.	0.	0.	0.	0.	0.	0.	87017.
40	0.	0.	0.	0.	0.	0.	0.	0.	0.
41	393.	11581.	902.	11590.	5009.	12784.	3525.	2019.	2478.
42	2726.	64175.	2687.	197823.	54181.	102864.	7871.	9589.	46918.
43	0.	0.	0.	0.	0.	0.	0.	0.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	0.	0.	0.	0.	0.	0.	0.	0.	0.
46	0.	3246.	0.	0.	0.	0.	0.	0.	5865.
47	383.	891.	45.	2828.	3631.	7468.	1862.	1103.	1557.
48	0.	12444.	770.	3803.	520.	26082.	12731.	33247.	123461.
49	302.	10382.	73.	402.	154.	5435.	1807.	942.	1536.
50	115.	2647.	135.	260.	45.	3597.	952.	448.	3398.
51	0.	0.	0.	0.	0.	0.	0.	0.	0.
52	0.	0.	0.	0.	0.	0.	0.	0.	0.
53	0.	205.	0.	151.	99.	4206.	324.	250.	14100.
54	0.	0.	0.	0.	0.	0.	0.	0.	0.
55	0.	10152.	936.	1235.	427.	12620.	1501.	1044.	1143.
56	0.	0.	0.	0.	0.	0.	0.	0.	0.
57	0.	0.	0.	0.	0.	0.	0.	0.	0.
58	0.	744.	36.	49.	6.	108.	46.	172.	246.
59	0.	0.	0.	0.	0.	0.	0.	0.	0.
60	0.	0.	0.	0.	0.	0.	0.	0.	0.
61	0.	5624.	0.	0.	0.	0.	0.	0.	0.
62	0.	0.	0.	0.	0.	0.	0.	0.	1811.
63	0.	0.	0.	0.	0.	0.	0.	49701.	445.
64	34402.	2663.	190.	2292.	234.	3692.	1342.	23322.	1935.
65	23176.	394576.	16370.	73105.	26592.	372289.	120653.	185203.	398041.
66	6023.	21916.	920.	17520.	6036.	29039.	7557.	161761.	57348.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	7770.	55202.	3534.	18865.	7810.	190343.	17217.	52755.	317054.
69	124557.	349725.	22595.	184032.	84423.	395803.	134337.	296531.	348401.
70	11068.	53690.	2670.	17220.	7630.	61353.	22595.	120330.	116400.
71	27381.	53028.	4190.	40039.	15798.	37660.	34139.	462063.	124933.
72	4076.	16003.	1190.	7052.	2507.	9866.	4692.	17230.	7810.
73	9609.	38731.	2134.	24859.	12271.	88364.	16244.	261980.	92310.
74	0.	0.	0.	0.	0.	1922.	0.	0.	26734.
75	3103.	68190.	3654.	5314.	2755.	7410.	3191.	12636.	16435.
76	69.	280.	22.	140.	45.	193.	102.	454.	100.
77	2418.	8656.	490.	3616.	1559.	9752.	3827.	13855.	11302.
78	3530.	3009.	405.	1816.	1420.	10322.	3101.	89310.	11700.
79	713.	5633.	231.	598.	236.	12780.	325.	1863.	6709.

ROW \ COL.	28	29	30	31	32	33	34	35	36
1	0.	4488.	0.	0.	0.	52732.	0.	0.	0.
2	0.	2306.	0.	0.	0.	0.	0.	0.	3826.
3	0.	0.	0.	0.	0.	934.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	800.	0.	0.	0.	0.	8824.
6	0.	0.	0.	0.	0.	0.	0.	93.	3660.
7	23390.	3019.	128.	9820.	9247.	2186.	0.	2822.	63099.
8	0.	0.	0.	8759386.	0.	0.	0.	0.	0.
9	0.	3488.	466.	46841.	3777.	0.	0.	24663.	448346.
10	0.	0.	0.	0.	7084.	805.	0.	606.	9837.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	27528.	1558.	681.	22140.	4856.	73.	345.	735.	2143.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	16651.	142563.	57587.	8852.	347.	209588.	705.	0.	4330.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	2949.	0.	210.	0.	137714.	0.	76193.	0.	11550.
17	0.	0.	0.	0.	443602.	0.	40913.	43.	987.
18	1935.	2304.	732.	2643.	7077.	0.	5625.	2641.	0.
19	0.	0.	0.	0.	859.	1360.	0.	0.	2015.
20	660.	0.	0.	0.	3433.	0.	24984.	29821.	7735.
21	910.	0.	0.	0.	0.	1867.	969.	11715.	11084.
22	0.	0.	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.	0.	0.
24	211374.	55100.	12070.	51381.	14800.	2658.	25570.	3984.	156375.
25	23049.	207783.	19911.	27018.	44700.	1231.	37660.	166989.	71285.
26	22604.	392560.	13532.	193803.	113807.	1581.	32359.	13746.	60364.
27	1206406.	613167.	398948.	447247.	289890.	62844.	1005.	84143.	145413.
28	113254.	600.	233184.	0.	908380.	0.	2026.	0.	81612.
29	37924.	432959.	11900.	35704.	4056.	22775.	960.	3252.	45431.
30	0.	8158.	2947.	0.	216.	18.	0.	52.	3607.
31	18402.	21784.	8422.	1200627.	13695.	3584.	747.	6708.	59480.
32	8084.	61528.	5228.	6442.	210256.	4142.	177795.	8137.	38650.
33	0.	0.	0.	0.	0.	126023.	653679.	0.	0.
34	19.	0.	16.	50.	0.	0.	261184.	9.	32.
35	0.	136918.	0.	646.	42981.	0.	132.	105952.	343.
36	0.	11829.	7853.	21480.	8920.	7386.	7257.	59277.	934620.
37	0.	0.	10302.	0.	7923.	0.	0.	0.	23732.
38	0.	0.	500.	0.	1997.	161.	304.	4499.	10782.
39	3845.	108428.	87798.	115134.	0.	0.	0.	0.	0.
40	0.	0.	0.	0.	0.	0.	0.	0.	0.
41	3398.	21862.	667.	1578.	3806.	573.	1520.	8928.	6067.
42	4071.	53972.	2271.	183910.	69480.	1329.	23287.	8351.	38378.
43	0.	0.	0.	0.	0.	0.	0.	0.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	0.	0.	0.	0.	0.	0.	0.	0.	0.
46	0.	0.	0.	0.	0.	0.	0.	0.	0.
47	6075.	654.	129.	2422.	2981.	619.	197.	3144.	3347.
48	3717.	1808.	0.	0.	0.	0.	0.	0.	900.
49	2444.	113.	302.	3588.	2478.	111.	240.	2815.	994.
50	993.	0.	141.	183.	17501.	101.	35.	1456.	1287.
51	0.	0.	0.	0.	0.	0.	0.	0.	0.
52	0.	0.	0.	0.	0.	0.	0.	0.	0.
53	2680.	198.	0.	2315.	1688.	516.	3.	1061.	5613.
54	0.	0.	0.	0.	0.	0.	0.	0.	0.
55	1675.	165.	279.	322.	6937.	818.	519.	1517.	26501.
56	0.	0.	0.	0.	0.	0.	0.	0.	0.
57	0.	0.	0.	0.	0.	0.	0.	0.	0.
58	29.	30.	35.	308.	0.	9.	14.	5.	257.
59	0.	0.	0.	0.	0.	0.	0.	0.	0.
60	0.	0.	0.	0.	0.	0.	0.	0.	0.
61	0.	0.	0.	0.	0.	0.	0.	0.	0.
62	781.	5301.	215.	767.	2161.	282.	0.	953.	2478.
63	184.	218.	70.	252.	625.	75.	0.	187.	582.
64	2695.	2876.	1333.	568.	0.	493.	2935.	893.	3943.
65	134403.	122316.	53556.	813925.	145014.	20497.	36945.	51144.	454794.
66	11006.	12121.	7267.	19738.	19160.	2622.	9024.	6448.	24682.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	36599.	20206.	6413.	236646.	67815.	6894.	9970.	80358.	210816.
69	81009.	173307.	78440.	166290.	229155.	23031.	33686.	77318.	217964.
70	28940.	49870.	13149.	104307.	36286.	5294.	22751.	20633.	71893.
71	22758.	48892.	20823.	127180.	66500.	2588.	27662.	15482.	52584.
72	2801.	3048.	990.	0.	8443.	1132.	9073.	3787.	10546.
73	32041.	668411.	16031.	140654.	58051.	3889.	66235.	22665.	55091.
74	12429.	0.	0.	5378.	0.	0.	0.	1967.	0.
75	1274.	2438.	2669.	17386.	2171.	571.	1011.	1551.	24465.
76	73.	0.	11.	0.	119.	22.	170.	77.	166.
77	4023.	6381.	1833.	15953.	6914.	927.	3609.	2339.	7840.
78	21226.	13852.	3975.	30198.	4776.	4582.	6981.	5608.	5967.
79	1320.	1367.	393.	6279.	1771.	327.	255.	891.	2669.

COL. ROW	37	38	39	40	41	42	43	44	45
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	3284.	0.
5	1020439.	16861.	0.	0.	0.	26000.	0.	0.	0.
6	2954.	792129.	0.	0.	0.	6263.	0.	0.	0.
7	494085.	10378.	364.	1575.	1172.	14078.	2319.	1968.	1774.
8	0.	0.	0.	0.	0.	0.	0.	0.	0.
9	55156.	4152.	0.	844.	0.	843.	0.	281.	235.
10	6628.	197.	0.	0.	0.	219.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	121557.	2228.	392.	6126.	1786.	2729.	581.	1264.	681.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	6085.	82.	0.	0.	0.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	15210.	0.	0.	0.	4054.	99.	146.	64.
17	695.	4461.	0.	0.	0.	5355.	369.	0.	0.
18	14111.	4824.	0.	7326.	3955.	7426.	1458.	1697.	2204.
19	1616.	1649.	0.	0.	0.	0.	0.	0.	0.
20	11349.	8850.	702.	8594.	14301.	47241.	380.	8177.	3076.
21	2885.	0.	986.	7594.	1656.	2314.	0.	187.	0.
22	0.	0.	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.	1330.	0.
24	46315.	26280.	3980.	12886.	15695.	5942.	1616.	1007.	2303.
25	10169.	5101.	27599.	30611.	315410.	35719.	11095.	4969.	1839.
26	72211.	40282.	15785.	44135.	119710.	51412.	15493.	47562.	22900.
27	142819.	78720.	0.	17692.	8998.	68877.	99.	1864.	1983.
28	4029.	113979.	1981.	1499.	12424.	4007.	995.	360.	1339.
29	35755.	9738.	4929.	1929.	1868.	431.	1001.	1374.	1512.
30	13383.	6731.	35186.	20639.	19006.	10290.	2218.	9602.	4122.
31	143574.	40066.	5456.	43166.	24078.	31210.	8660.	9007.	12902.
32	56543.	11594.	30745.	7926.	18764.	41604.	9772.	84116.	45132.
33	0.	0.	786.	883.	145.	335.	305.	2673.	337.
34	83.	0.	7.	0.	0.	0.	19.	0.	23.
35	985.	244.	16.	30242.	2766.	980.	773.	244.	88.
36	290283.	41037.	4012.	43877.	27843.	52456.	14195.	12515.	15890.
37	3967485.	15940.	929950.	1928324.	699504.	925498.	223325.	347095.	466089.
38	223278.	3064491.	16885.	581113.	242042.	380763.	67929.	13733.	13217.
39	0.	0.	3241.	0.	0.	0.	0.	0.	0.
40	0.	0.	0.	156399.	0.	0.	0.	0.	28315.
41	107124.	84693.	19084.	117183.	105793.	89533.	42929.	71973.	27726.
42	291824.	86782.	6375.	210410.	60813.	256193.	2975.	8960.	27095.
43	0.	0.	0.	695.	4597.	0.	210657.	119723.	73884.
44	0.	0.	0.	0.	0.	0.	0.	91709.	1443.
45	0.	0.	0.	0.	0.	0.	0.	0.	181722.
46	0.	0.	0.	0.	0.	0.	2395.	0.	0.
47	107057.	60598.	25849.	35100.	21836.	160602.	36578.	38014.	40637.
48	12275.	0.	0.	56.	748.	566.	0.	0.	0.
49	43992.	23382.	14430.	62542.	0.	5964.	46589.	138466.	156306.
50	147362.	32024.	10228.	25472.	10832.	6147.	74775.	35519.	11154.
51	0.	0.	0.	0.	0.	0.	0.	0.	0.
52	0.	0.	0.	10299.	0.	0.	0.	0.	0.
53	76614.	7901.	2098.	72588.	1990.	19869.	15793.	10497.	28158.
54	0.	0.	0.	0.	625.	0.	0.	4473.	0.
55	2211.	39202.	812.	6545.	5612.	3947.	691.	1362.	1194.
56	0.	0.	0.	0.	0.	0.	0.	0.	0.
57	0.	0.	0.	0.	930.	2423.	0.	0.	0.
58	98.	63.	18.	195.	44.	91.	40316.	17547.	2977.
59	0.	0.	0.	0.	0.	0.	2747.	9849.	986.
60	0.	0.	0.	0.	735.	0.	0.	0.	0.
61	0.	0.	0.	0.	0.	0.	0.	0.	0.
62	4744.	1857.	377.	66072.	136.	1966.	1465.	2622.	151.
63	1362.	412.	115.	654.	369.	589.	97.	159.	168.
64	11322.	6752.	1535.	5107.	496.	3052.	3113.	1196.	1535.
65	963416.	197481.	53018.	145351.	56604.	109512.	26948.	37408.	42522.
66	65153.	26077.	1980.	31642.	8272.	17307.	4768.	5369.	8805.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	433213.	213229.	13751.	47538.	32910.	58257.	8166.	11978.	17584.
69	657030.	307284.	80999.	294478.	105285.	238917.	57141.	96408.	104409.
70	136349.	57793.	12732.	63998.	28985.	45658.	12734.	14620.	20222.
71	67905.	35184.	8167.	44069.	28912.	32910.	8467.	11292.	13944.
72	20463.	6984.	1758.	9927.	5390.	10184.	2033.	2422.	3006.
73	112289.	41495.	9744.	43481.	19853.	50789.	12338.	17995.	17389.
74	16485.	4721.	0.	2040.	0.	0.	13180.	0.	0.
75	5518.	5379.	772.	16565.	2624.	6453.	541.	2309.	2298.
76	454.	150.	34.	144.	101.	182.	45.	43.	54.
77	19106.	9035.	2122.	8113.	3674.	6633.	2073.	2344.	2954.
78	14380.	4112.	1415.	7669.	3666.	6561.	1909.	4595.	1986.
79	13440.	2802.	189.	1740.	994.	1628.	300.	742.	401.

COL. ROW	46	47	48	49	50	51	52	53	54
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	0.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	3192.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	0.	0.	33.	631.	112.	0.	771.	1564.	1074.
8	0.	0.	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	0.	10323.	0.	0.	0.	0.	0.
10	0.	0.	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	341.	6761.	1843.	3839.	6139.	1207.	2226.	5909.	1658.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	0.	0.	0.	0.	0.	0.	0.	0.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	1775.	0.	2840.	1217.	0.	0.	68.	878.	9017.
17	0.	0.	0.	0.	0.	0.	0.	1466.	247.
18	825.	4423.	2592.	3640.	2099.	1504.	1607.	4465.	2382.
19	0.	0.	0.	0.	0.	0.	0.	0.	0.
20	134.	5042.	11710.	4438.	0.	461.	5125.	6075.	4316.
21	0.	0.	203.	0.	0.	607.	12346.	121.	20261.
22	0.	0.	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.	0.	0.
24	495.	0.	3659.	10374.	49.	9190.	9330.	40189.	4132.
25	130.	783.	6390.	5435.	0.	2031.	20007.	17980.	33075.
26	8241.	13727.	89430.	15843.	4579.	30898.	6885.	18337.	246609.
27	0.	2462.	3462.	4893.	0.	422.	15569.	15710.	17056.
28	508.	756.	1396.	121.	0.	1730.	2319.	35377.	8703.
29	842.	1517.	1301.	1019.	912.	0.	404.	463.	477.
30	3037.	166.	589.	2014.	0.	1693.	12036.	14917.	20112.
31	3670.	20104.	14496.	13832.	16550.	2114.	6376.	15668.	3331.
32	23651.	984.	20324.	12819.	514.	22249.	26952.	32036.	119501.
33	66.	217.	3319.	883.	501.	96.	278.	500.	778.
34	6.	24.	12.	7.	2.	0.	0.	17.	4625.
35	219.	403.	49.	48.	29.	416.	3011.	3867.	3754.
36	4079.	26855.	8712.	30042.	25287.	4007.	13900.	39448.	27394.
37	125329.	263275.	218962.	385746.	115552.	42654.	146329.	286347.	250923.
38	11005.	72406.	105631.	86495.	117211.	41182.	120938.	321463.	140469.
39	0.	0.	0.	0.	0.	0.	0.	0.	0.
40	8543.	3284.	19751.	22635.	0.	0.	5011.	10996.	7663.
41	14762.	29293.	24731.	37392.	2444.	25082.	86039.	69364.	172513.
42	11067.	12720.	29000.	48300.	17949.	9262.	51986.	29135.	110562.
43	9177.	0.	198.	8041.	0.	0.	2636.	6875.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	267.	0.	0.	0.	0.	0.	0.	0.	0.
46	45275.	3296.	45.	0.	0.	0.	0.	0.	0.
47	13020.	241230.	27905.	56407.	22960.	21284.	3495.	50647.	25740.
48	0.	1403.	133018.	0.	0.	0.	0.	0.	0.
49	56676.	110868.	113862.	287634.	2643.	11377.	8111.	36348.	31263.
50	18401.	4601.	6974.	11365.	110909.	3496.	1815.	6511.	2964.
51	125.	0.	1046.	0.	0.	181420.	0.	0.	0.
52	0.	0.	0.	0.	0.	0.	119706.	0.	61305.
53	58241.	95868.	89564.	143712.	6280.	47872.	219309.	356197.	118104.
54	0.	0.	0.	0.	0.	0.	0.	0.	39995.
55	1351.	488.	445.	1950.	939.	7617.	16000.	39679.	25721.
56	0.	0.	0.	0.	0.	4.	0.	0.	0.
57	0.	0.	0.	477.	0.	90596.	123.	92159.	0.
58	2811.	32.	50.	577.	24.	0.	29.	0.	9.
59	0.	0.	0.	0.	0.	0.	0.	0.	0.
60	0.	0.	0.	0.	0.	0.	0.	0.	0.
61	0.	0.	0.	0.	0.	0.	0.	0.	0.
62	0.	1518.	2544.	16198.	968.	667.	28661.	3880.	107180.
63	65.	491.	133.	455.	128.	124.	134.	383.	159.
64	529.	2041.	1139.	729.	380.	170.	2789.	1581.	1652.
65	13631.	29516.	27247.	51343.	15103.	14590.	35312.	59067.	54274.
66	3461.	33387.	25401.	36390.	10789.	1473.	7165.	19777.	14002.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	3913.	23392.	13006.	22563.	12275.	7200.	10537.	36306.	20131.
69	50138.	113573.	98389.	182766.	48951.	107025.	132541.	144489.	142589.
70	8831.	30017.	17201.	22878.	12711.	8921.	18354.	21187.	8965.
71	10206.	71237.	23428.	25034.	23319.	16676.	29691.	35802.	14497.
72	1181.	6360.	3698.	4782.	3029.	2137.	2171.	6176.	3300.
73	9585.	26432.	18709.	28442.	10600.	19959.	12294.	37667.	71744.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	605.	2532.	3603.	2450.	1895.	420.	2299.	2264.	595.
76	20.	140.	69.	101.	59.	28.	26.	146.	57.
77	1015.	3830.	2503.	3733.	1680.	1787.	2156.	5027.	3424.
78	1386.	2856.	2314.	3694.	1837.	2520.	2238.	12531.	7323.
79	73.	423.	448.	586.	274.	0.	347.	836.	622.

COL. ROW	55	56	57	58	59	60	61	62	63
1	0.	0.	0.	0.	0.	0.	0.	0.	0.
2	0.	0.	0.	0.	0.	0.	0.	4995.	0.
3	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	0.	0.	0.	0.
5	0.	0.	0.	2008.	0.	0.	0.	0.	0.
6	2001.	0.	0.	0.	0.	0.	0.	1526.	0.
7	0.	1806.	418.	0.	13776.	2301.	2139.	0.	2414.
8	0.	0.	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	0.	251.	0.	0.	156.	0.	0.
10	0.	0.	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	126.	7773.	1371.	96.	68091.	21119.	787.	533.	482.
13	0.	0.	0.	0.	0.	202011.	0.	0.	0.
14	0.	0.	0.	0.	0.	0.	0.	12683.	0.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	0.	853.	0.	314.	50999.	1635.	1928.	27472.	436.
17	0.	0.	0.	0.	89280.	5029.	2421.	5507.	312.
18	2165.	4432.	3764.	1459.	9712.	11644.	3938.	3060.	1307.
19	0.	0.	0.	0.	147654.	0.	1435.	0.	0.
20	980.	10357.	278.	0.	8425.	19981.	87156.	951.	95.
21	0.	0.	0.	0.	0.	0.	0.	1047.	0.
22	0.	143668.	13459.	0.	0.	0.	24095.	0.	0.
23	0.	0.	0.	0.	4085.	16359.	13950.	0.	0.
24	5705.	25284.	13281.	1224.	82553.	4056.	5387.	13979.	50049.
25	11061.	30900.	15103.	10936.	21094.	8091.	1196.	28395.	10151.
26	13633.	95389.	4977.	19475.	266694.	24633.	12426.	18157.	55941.
27	11150.	5017.	32941.	35113.	38570.	13990.	7873.	7407.	92807.
28	43766.	42543.	21214.	7912.	25365.	7181.	27399.	7287.	792.
29	328.	2048.	114.	9.	18397.	5116.	2190.	0.	0.
30	11043.	2169.	1522.	9.	88346.	10239.	30015.	1567.	244.
31	3583.	6166.	4120.	1578.	44486.	30158.	15709.	3856.	3744.
32	33681.	59203.	17179.	75743.	624273.	81545.	31457.	40684.	10763.
33	0.	983.	0.	83.	6358.	0.	1521.	1507.	97.
34	0.	0.	0.	1.	8.	75.	0.	1836.	486.
35	68112.	24834.	91833.	481.	234701.	1505.	18892.	10447.	16462.
36	12367.	18284.	18163.	15108.	56709.	45085.	30227.	12176.	4283.
37	137932.	53260.	46899.	42260.	1933862.	393695.	414573.	63431.	7347.
38	97783.	108969.	123548.	147907.	219192.	352295.	57432.	144627.	42088.
39	0.	0.	0.	0.	0.	0.	0.	9105.	0.
40	0.	2498.	193.	0.	6332.	688.	154152.	2427.	0.
41	58855.	113580.	65486.	37234.	694517.	250217.	14226.	66992.	8525.
42	34428.	72709.	36661.	3814.	827797.	121196.	71827.	40708.	15330.
43	0.	0.	0.	0.	52976.	1784.	99797.	0.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	0.	0.	0.	0.	0.	0.	17065.	0.	0.
46	0.	0.	0.	0.	0.	0.	12121.	0.	0.
47	10767.	23901.	15525.	21388.	241590.	241419.	16248.	42986.	6451.
48	0.	0.	0.	0.	0.	0.	0.	0.	0.
49	198.	1506.	999.	27348.	91322.	121572.	52698.	13366.	0.
50	3370.	4612.	2742.	2715.	132525.	119444.	11785.	17980.	0.
51	0.	0.	0.	0.	0.	753.	0.	23826.	0.
52	0.	3560.	0.	0.	25373.	696.	8780.	333.	0.
53	33053.	51260.	7367.	11603.	18140.	40517.	121898.	70675.	12787.
54	0.	0.	0.	0.	0.	0.	19396.	0.	0.
55	95758.	65383.	10578.	27346.	90620.	20248.	11629.	10271.	3806.
56	0.	348016.	491.	0.	111407.	312285.	5259.	14320.	0.
57	0.	1062052.	162519.	10524.	14580.	68699.	293.	98825.	0.
58	44750.	0.	0.	61007.	324011.	47894.	3863.	146.	0.
59	0.	0.	0.	5817.	6807635.	41522.	24478.	0.	0.
60	0.	0.	0.	0.	0.	2441590.	0.	0.	0.
61	0.	0.	0.	0.	1353.	0.	255873.	0.	0.
62	488.	9096.	2030.	869.	92383.	162560.	4465.	227473.	777.
63	79.	294.	339.	95.	801.	27059.	323.	2530.	87053.
64	6971.	1626.	2368.	650.	19072.	7148.	7494.	14966.	565.
65	29378.	68847.	28678.	19385.	418846.	108858.	62542.	34045.	21898.
66	5526.	19229.	7470.	4677.	43536.	57982.	9294.	15958.	4191.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	10612.	15408.	20650.	7870.	103024.	69633.	22236.	12258.	4545.
69	146203.	208043.	154734.	51810.	665316.	220830.	173595.	140436.	55021.
70	9060.	19072.	12705.	5391.	81138.	33745.	17431.	17325.	7545.
71	20338.	30886.	57080.	9677.	53174.	67212.	14919.	34462.	18267.
72	2973.	6054.	5384.	1921.	14017.	0.	5607.	4818.	1817.
73	14894.	50571.	22218.	12651.	310749.	25330.	17782.	30429.	33021.
74	0.	1961.	2145.	0.	11500.	12586.	0.	0.	1976.
75	493.	1200.	45.	632.	7321.	968.	3158.	737.	783.
76	57.	106.	116.	16.	281.	0.	113.	87.	13.
77	2274.	6017.	2854.	1456.	22412.	11352.	3765.	3535.	1569.
78	3142.	14478.	9171.	3513.	38263.	11727.	2532.	3738.	3066.
79	392.	527.	501.	328.	3987.	1994.	790.	467.	119.

COL. ROW	64	65	66	67	68	69	70	71	72
1	0.	237.	0.	0.	0.	0.	0.	0.	0.
2	7780.	7450.	0.	0.	0.	0.	0.	23040.	0.
3	1777.	1329.	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	0.	0.	159765.	0.	0.	0.
5	0.	0.	0.	0.	0.	0.	0.	0.	0.
6	0.	0.	0.	0.	0.	0.	0.	0.	0.
7	80.	26917.	0.	0.	636912.	0.	5974.	0.	0.
8	0.	0.	0.	0.	1124711.	0.	0.	0.	0.
9	298.	1325.	0.	0.	0.	0.	0.	4660.	0.
10	28.	828.	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.	0.	0.
12	14288.	1375950.	281082.	7356.	1220486.	711969.	49109.	6505818.	14733.
13	0.	0.	0.	0.	0.	0.	0.	0.	0.
14	6945.	100182.	0.	0.	0.	2425.	0.	0.	13203.
15	0.	0.	0.	0.	0.	0.	0.	0.	0.
16	97042.	6707.	691.	0.	0.	0.	0.	0.	125907.
17	31497.	15865.	3146.	1737.	0.	16756.	21208.	0.	22998.
18	3311.	4403.	0.	0.	0.	0.	325.	0.	96574.
19	1434.	14605.	4461.	2478.	0.	21811.	30167.	0.	125096.
20	83140.	8807.	150.	0.	0.	4140.	0.	0.	4777.
21	1745.	17185.	0.	0.	0.	90292.	0.	0.	0.
22	0.	0.	0.	0.	0.	0.	0.	0.	10563.
23	0.	0.	0.	0.	0.	0.	0.	0.	3171.
24	107128.	33607.	0.	8263.	9697.	444665.	101646.	0.	144763.
25	207822.	8200.	0.	1971.	0.	355237.	24195.	0.	18698.
26	73274.	160148.	136802.	23614.	50645.	1203098.	691067.	255150.	104672.
27	32768.	36187.	248.	0.	111.	0.	0.	19831.	98083.
28	118775.	108.	0.	0.	0.	0.	0.	0.	0.
29	4428.	11698.	308.	42.	434.	0.	11112.	0.	175844.
30	39948.	39600.	0.	0.	0.	0.	0.	0.	101.
31	14210.	1551162.	13598.	1948.	256068.	706415.	90841.	321013.	142425.
32	114797.	260558.	4729.	848.	8859.	160173.	52870.	38662.	68303.
33	32501.	3262.	0.	0.	0.	0.	0.	0.	0.
34	25865.	384.	234.	39.	156.	3694.	974.	244.	6676.
35	25530.	5244.	0.	0.	0.	102550.	0.	0.	3097.
36	7883.	3856.	0.	0.	23608.	43339.	0.	15.	51878.
37	130161.	38669.	0.	0.	2004.	0.	0.	0.	0.
38	245267.	49212.	22535.	0.	0.	0.	0.	0.	6296.
39	0.	0.	0.	0.	0.	3584.	0.	0.	0.
40	0.	535.	0.	0.	0.	0.	0.	0.	0.
41	41342.	15291.	0.	0.	0.	20115.	0.	0.	0.
42	78078.	42109.	2881.	0.	183683.	9666.	0.	0.	28022.
43	0.	81601.	0.	0.	0.	0.	0.	0.	0.
44	0.	0.	0.	0.	0.	0.	0.	0.	0.
45	0.	0.	0.	0.	0.	0.	0.	0.	0.
46	0.	12797.	0.	0.	0.	0.	0.	0.	0.
47	1493.	23616.	100.	0.	2241.	0.	0.	0.	0.
48	0.	0.	0.	0.	0.	0.	0.	0.	573.
49	1985.	14441.	0.	0.	0.	0.	0.	0.	0.
50	10307.	7248.	0.	0.	0.	13660.	0.	0.	0.
51	0.	2639.	0.	0.	8538.	8006.	4948.	0.	0.
52	479.	2111.	0.	0.	0.	4987.	0.	0.	38375.
53	27351.	30920.	408.	0.	9794.	0.	0.	0.	6637.
54	0.	0.	0.	0.	0.	0.	0.	0.	116729.
55	14123.	7123.	0.	0.	2705.	3671.	0.	0.	7442.
56	0.	17405.	113066.	34523.	3598.	2652.	0.	0.	9856.
57	7042.	24421.	5114.	0.	0.	0.	0.	0.	227375.
58	73.	72636.	4222.	0.	395.	22247.	1881.	1141.	3538.
59	0.	89525.	0.	0.	16.	167969.	0.	0.	0.
60	0.	167609.	0.	0.	0.	0.	0.	0.	0.
61	0.	297543.	4236.	0.	1155.	0.	9336.	0.	8700.
62	589.	25650.	48.	0.	0.	0.	0.	0.	69373.
63	0.	20.	3.	0.	0.	0.	0.	0.	101921.
64	296947.	48474.	7345.	9041.	6294.	45692.	22689.	0.	260043.
65	78117.	2118467.	13936.	2542.	436719.	364992.	235242.	29659.	87182.
66	25582.	276262.	76272.	61248.	55023.	1044594.	410822.	198098.	61758.
67	0.	0.	0.	0.	0.	0.	0.	0.	0.
68	20574.	147436.	50999.	6219.	3830234.	1976109.	122692.	254422.	232452.
69	313702.	1011028.	39465.	18189.	265166.	1632881.	254071.	187673.	515654.
70	39140.	696411.	44663.	12193.	124013.	1600399.	5374161.	1509855.	189085.
71	65777.	1036073.	130706.	77571.	64725.	5199167.	2047594.	1573780.	552704.
72	9493.	0.	0.	0.	0.	230575.	0.	0.	353547.
73	52892.	416115.	119800.	52124.	217053.	3750660.	719897.	590194.	208826.
74	0.	0.	0.	0.	0.	0.	0.	0.	0.
75	6524.	821341.	12398.	1668.	26218.	848730.	82461.	51497.	114150.
76	149.	25022.	2900.	300614.	0.	97879.	5007.	0.	0.
77	5552.	31383.	8537.	1493.	16680.	97491.	140281.	14036.	11637.
78	4941.	42044.	23392.	851.	52066.	423044.	322753.	252415.	3991.
79	1147.	56062.	3590.	140.	6366.	51922.	7319.	2684.	18550.

COL. ROW	73	74	75	76	77	78	79
1	0.	0.	0.	10865.	4552.	1737.	1737.
2	0.	0.	0.	0.	5412.	621569.	0.
3	0.	0.	0.	0.	0.	0.	0.
4	0.	0.	0.	3146.	0.	0.	43.
5	0.	0.	0.	0.	0.	1651.	0.
6	0.	0.	0.	0.	0.	0.	0.
7	31432.	0.	10493.	0.	213.	33820.	0.
8	0.	0.	0.	0.	0.	0.	0.
9	0.	0.	0.	0.	0.	357.	0.
10	0.	0.	0.	0.	0.	0.	0.
11	0.	0.	0.	0.	0.	0.	0.
12	21348.	0.	99567.	117599.	661023.	1572.	261318.
13	0.	0.	0.	0.	0.	1000.	0.
14	0.	0.	0.	0.	170998.	259009.	0.
15	0.	0.	0.	0.	176.	0.	0.
16	0.	0.	0.	0.	2458.	0.	0.
17	14691.	1747.	16261.	3179.	22722.	0.	1680.
18	4569.	433.	1146.	0.	38344.	0.	1578.
19	22056.	2570.	17753.	5196.	33308.	1798.	0.
20	3200.	0.	0.	0.	3122.	0.	0.
21	0.	0.	0.	0.	0.	0.	0.
22	0.	0.	0.	0.	0.	0.	0.
23	0.	0.	0.	0.	0.	0.	0.
24	72663.	6777.	3286.	3238.	84920.	23939.	103.
25	8512.	1620.	940.	788.	20260.	4157.	0.
26	89770.	468.	34021.	118766.	383276.	30277.	7858.
27	14347.	0.	43.	0.	4024.	0.	22760.
28	4500.	0.	0.	0.	0.	0.	37.
29	29259.	5607.	9344.	869.	587580.	0.	2464.
30	0.	0.	60920.	0.	0.	0.	13.
31	112281.	323.	27209.	3064.	70592.	2515.	474.
32	71841.	17152.	273485.	1518.	64597.	750.	0.
33	0.	0.	0.	0.	0.	0.	0.
34	507.	0.	49.	9613.	2553.	718.	0.
35	0.	0.	96076.	0.	6203.	0.	0.
36	233.	0.	39514.	0.	77.	10919.	0.
37	5000.	0.	0.	0.	0.	0.	1384.
38	25473.	0.	0.	0.	0.	0.	0.
39	0.	0.	0.	0.	0.	0.	0.
40	0.	0.	0.	0.	0.	0.	0.
41	0.	0.	0.	0.	20418.	0.	0.
42	3812.	0.	114822.	0.	303.	0.	0.
43	108287.	0.	0.	0.	0.	0.	0.
44	142748.	0.	0.	0.	0.	0.	0.
45	18338.	0.	0.	0.	0.	0.	0.
46	0.	0.	0.	0.	0.	0.	0.
47	19014.	930.	967.	0.	0.	0.	0.
48	0.	0.	0.	0.	0.	0.	0.
49	0.	0.	0.	0.	0.	0.	0.
50	430.	0.	107915.	0.	103.	647.	551.
51	87680.	0.	0.	0.	2979.	0.	0.
52	88739.	1505.	0.	0.	0.	0.	0.
53	97.	0.	6138.	0.	0.	0.	0.
54	0.	0.	0.	0.	0.	0.	0.
55	115.	0.	33414.	0.	30.	113.	0.
56	0.	0.	0.	0.	10394.	0.	0.
57	0.	0.	0.	0.	0.	0.	0.
58	4829.	0.	116669.	0.	11534.	333.	0.
59	5268.	0.	1131749.	0.	1699.	8248.	6825.
60	0.	0.	0.	0.	0.	0.	0.
61	18854.	5714.	9361.	894.	19309.	0.	0.
62	0.	0.	15950.	0.	258460.	0.	0.
63	198160.	0.	0.	23716.	70968.	0.	0.
64	78092.	4933.	1821.	87225.	32504.	0.	51.
65	120125.	2007.	74462.	22256.	119934.	703330.	10965.
66	285696.	2000.	55351.	30181.	191199.	3405.	6485.
67	7190.	0.	0.	0.	0.	0.	0.
68	251707.	0.	152052.	41421.	420107.	15721.	3444.
69	443119.	11628.	673054.	69314.	420033.	44980.	6439.
70	292198.	5000.	206248.	126256.	264655.	0.	0.
71	929515.	11637.	310306.	280876.	1564472.	5670.	9153.
72	0.	0.	0.	0.	110360.	0.	6640.
73	541235.	20000.	130033.	110842.	514693.	27443.	15116.
74	0.	0.	0.	0.	39215.	1260.	0.
75	123048.	0.	133548.	0.	48819.	29567.	739.
76	0.	5000.	0.	1327366.	83911.	0.	0.
77	3684.	516.	7617.	5394.	296086.	0.	10.
78	689046.	0.	3267.	2489.	15818.	2632.	648.
79	7656.	0.	609.	1192.	12331.	369.	75.

CONTROL TOTALS ** NPA **

1	23842352.
2	20735344.
3	1147388.
4	1557978.
5	765376.
6	1001905.
7	2730349.
8	9134314.
9	1431244.
10	323342.
11	52416000.
12	16874992.
13	3872057.
14	63151568.
15	5909499.
16	10085330.
17	2006639.
18	14101231.
19	2157340.
20	7714229.
21	412967.
22	3175923.
23	1389723.
24	9261048.
25	3519415.
26	12312526.
27	10875211.
28	3881191.
29	6200501.
30	1761577.
31	16504990.
32	6387139.
33	869626.
34	3040415.
35	2112954.
36	7157081.
37	17901728.
38	8650457.
39	2041241.
40	7442870.
41	3274019.
42	5934857.
43	1941932.
44	2185429.
45	2746395.
46	955707.
47	3396622.
48	2220224.
49	3361200.
50	1442602.
51	1657016.
52	1975747.
53	4577054.
54	3250140.
55	2090519.
56	5503085.
57	2472922.
58	1356380.
59	21846048.
60	11870805.
61	3444025.
62	3105906.
63	1427495.
64	4749407.
65	32800992.
66	8886570.
67	1534186.
68	20194416.
69	94350096.
70	25695456.
71	61934288.
72	11788499.
73	16962304.
74	5339304.
75	7843002.
76	5489100.
77	22677952.
78	3144265.
79	741900.

TABLE F-6

"COMMODITY TECHNOLOGY" MATRIX, UNITED STATES ECONOMY, 1958

$(A^D + M) = [a^D_{ij} + m_{ij}] \quad i,j = 1, \ldots, 79.$

COL. / ROW	1	2	3	4	5	6	7	8	9
1	0.16281	0.07791	0.0	0.16676	0.0	0.0	0.0	0.0	0.0
2	0.25897	0.03242	0.01336	0.02392	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.01657	0.0	0.0	0.0	0.0	0.0	0.0
4	0.02054	0.04083	0.01932	0.01576	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.08436	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.02375	0.23122	0.0	0.0	0.0
7	0.00021	0.00002	0.0	0.0	0.00605	0.00107	0.17234	0.00002	0.00075
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.02556	0.0
9	0.00003	0.00314	0.0	0.00000	0.0	0.0	0.00036	0.0	0.00600
10	0.0	0.00127	0.00007	0.0	0.0	0.0	0.00002	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00605	0.01272	0.00024	0.00134	0.00064	0.00088	0.00072	0.00042	0.00105
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.11547	0.00013	0.02702	0.00743	0.0	0.0	0.0	0.0	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.00034	0.0	0.0	0.00017	0.00186	0.00068	0.0	0.0
17	0.00020	0.00114	0.01259	0.00779	0.0	0.0	0.0	0.00023	0.00003
18	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
19	0.00024	0.00158	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.00007	0.00006	0.0	0.0	0.00771	0.00108	0.00668	0.00060	0.0
21	0.0	0.00467	0.0	0.00007	0.0	0.0	0.0	0.0	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.0	0.0	0.00889	0.00193	0.0	0.00051	0.00231	0.00038	0.00891
25	0.00053	0.00012	0.01020	0.00601	0.0	0.0	0.00054	0.00000	0.00175
26	0.00021	0.00071	0.00009	0.00000	0.0	0.00380	0.00027	0.00030	0.00034
27	0.00125	0.05388	0.00019	0.00031	0.01431	0.04205	0.01613	0.00516	0.01174
28	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
29	0.00104	0.0	0.0	0.00005	0.00001	0.00011	0.0	0.00000	0.00013
30	0.0	0.0	0.00200	0.0	0.0	0.00020	0.00032	0.00054	0.0
31	0.00177	0.04022	0.02064	0.00258	0.01361	0.00801	0.01114	0.00546	0.03211
32	0.00078	0.00710	0.00934	0.00180	0.00068	0.00380	0.00836	0.00335	0.02110
33	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
34	0.00003	0.00017	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
35	0.00008	0.0	0.0	0.0	0.0	0.0	0.0	0.00004	0.0
36	0.00004	0.00115	0.0	0.0	0.00090	0.00622	0.00232	0.00037	0.00034
37	0.0	0.0	0.0	0.0	0.02665	0.05136	0.00838	0.00032	0.01611
38	0.00003	0.00004	0.0	0.0	0.00211	0.00647	0.00616	0.00079	0.00123
39	0.00020	0.00058	0.0	0.0	0.0	0.0	0.0	0.0	0.0
40	0.0	0.0	0.0	0.0	0.00152	0.00043	0.00037	0.00065	0.00001
41	0.00075	0.0	0.0	0.0	0.00034	0.00059	0.00578	0.00060	0.00013
42	0.00112	0.00166	0.00077	0.00891	0.00104	0.00133	0.00519	0.00497	0.00054
43	0.0	0.0	0.00014	0.0	0.00051	0.00053	0.0	0.00161	0.00000
44	0.00020	0.00918	0.0	0.0	0.0	0.0	0.0	0.0	0.00002
45	0.0	0.0	0.0	0.0	0.03095	0.02799	0.03589	0.00416	0.05381
46	0.0	0.0	0.0	0.0	0.00000	0.00011	0.00370	0.0	0.01518
47	0.0	0.0	0.0	0.0	0.00006	0.00073	0.00361	0.0	0.00006
48	0.0	0.0	0.0	0.0	0.0	0.00001	0.0	0.00094	0.0
49	0.0	0.0	0.0	0.0	0.00024	0.00239	0.00177	0.00838	0.00379
50	0.00008	0.00013	0.0	0.0	0.00007	0.00009	0.00021	0.00008	0.00036
51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
53	0.0	0.0	0.0	0.0	0.00093	0.00485	0.00209	0.00357	0.00212
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
55	0.00003	0.00004	0.00011	0.0	0.00048	0.00050	0.00153	0.00007	0.00009
56	0.0	0.0	0.0	0.0	0.00166	0.0	0.0	0.0	0.0
57	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00121	0.0
58	0.00030	0.00097	0.0	0.0	0.00030	0.00016	0.00010	0.00012	0.00020
59	0.00097	0.00160	0.0	0.0	0.00070	0.00098	0.00207	0.00097	0.00437
60	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.0	0.00014	0.02179	0.0	0.00322	0.0	0.00415	0.0	0.00019
62	0.0	0.0	0.0	0.0	0.00025	0.00042	0.00005	0.00006	0.00022
63	0.0	0.0	0.0	0.0	0.00008	0.00007	0.00002	0.00000	0.00008
64	0.00001	0.00002	0.00252	0.00050	0.00001	0.00000	0.00146	0.00004	0.00041
65	0.01933	0.01186	0.00754	0.01079	0.03157	0.01450	0.00751	0.00871	0.00756
66	0.00193	0.00298	0.00352	0.00640	0.00202	0.00192	0.00070	0.00016	0.00178
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.00332	0.00724	0.00050	0.00113	0.02172	0.02912	0.02373	0.00630	0.02713
69	0.03598	0.04697	0.01992	0.00433	0.02286	0.02531	0.03481	0.01062	0.04433
70	0.00654	0.01092	0.03213	0.00498	0.00753	0.01691	0.00988	0.01108	0.01219
71	0.01131	0.07818	0.04249	0.01578	0.10465	0.03662	0.02212	0.15049	0.02904
72	0.0	0.0	0.0	0.0	0.00120	0.00107	0.00024	0.00005	0.00105
73	0.00103	0.03422	0.11088	0.00484	0.00747	0.00777	0.00486	0.03645	0.00990
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.00219	0.00220	0.0	0.0	0.0	0.0	0.00034	0.00150	0.00020
76	0.0	0.0	0.0	0.0	0.00003	0.00003	0.00000	0.00000	0.00002
77	0.00589	0.00052	0.00097	0.00088	0.00108	0.00106	0.00099	0.00098	0.00105
78	0.00015	0.00014	0.00099	0.00097	0.00097	0.00096	0.00098	0.0	0.00065
79	0.00002	0.00003	0.00010	0.00017	0.00010	0.00037	0.00023	0.00046	0.00106

COL. ROW	10	11	12	13	14	15	16	17	18
1	0.0	0.0	0.0	0.0	0.24223	0.0	0.01013	0.02850	0.0
2	0.0	0.00452	0.0	0.0	0.08113	0.18529	0.11868	0.00172	0.00046
3	0.0	0.0	0.0	0.0	0.00425	0.0	0.0	0.0	0.00950
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.00109	0.00000	0.0	0.0	0.00064	0.00022	0.00154	0.00066	0.00004
8	0.0	0.00000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.02359	0.01193	0.00777	0.0	0.00006	0.0	0.0	0.00000	0.0
10	0.08166	0.0	0.0	0.0	0.00011	0.0	0.00007	0.0	0.00002
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00054	0.00013	0.00006	0.00169	0.00354	0.00004	0.00063	0.00018	0.00042
13	0.0	0.00010	0.0	0.02094	0.0	0.0	0.0	0.0	0.0
14	0.00012	0.00032	0.0	0.0	0.17056	0.00594	0.00221	0.00803	0.0
15	0.0	0.0	0.0	0.0	0.0	0.19176	0.0	0.0	0.0
16	0.00045	0.0	0.0	0.0	0.00004	0.00019	0.35499	0.16512	0.27676
17	0.0	0.00007	0.00006	0.00002	0.00001	0.0	0.01119	0.10900	0.00532
18	0.0	0.0	0.0	0.00057	0.00062	0.0	0.0	0.0	0.17443
19	0.0	0.00001	0.00004	0.0	0.00156	0.0	0.00047	0.00004	0.01138
20	0.00044	0.06258	0.02476	0.00019	0.00005	0.00021	0.00016	0.0	0.0
21	0.0	0.0	0.0	0.00126	0.00157	0.00155	0.0	0.0	0.0
22	0.0	0.00566	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.00390	0.00098	0.0	0.0	0.0	0.0	0.0	0.0
24	0.00648	0.00617	0.00404	0.00110	0.00594	0.01168	0.00143	0.00665	0.00094
25	0.00102	0.0	0.0	0.03846	0.01414	0.01175	0.00794	0.00715	0.00619
26	0.00006	0.00167	0.00016	0.00182	0.01209	0.01645	0.00506	0.00412	0.00370
27	0.00989	0.00700	0.00419	0.00260	0.00273	0.00087	0.01657	0.00247	0.00296
28	0.0	0.0	0.0	0.0	0.00017	0.01789	0.09117	0.20285	0.01017
29	0.00032	0.0	0.0	0.00073	0.00251	0.00126	0.00229	0.00041	0.00008
30	0.0	0.00375	0.05211	0.00047	0.00009	0.00002	0.00036	0.00002	0.00002
31	0.01115	0.01882	0.02222	0.00243	0.00453	0.00045	0.00254	0.00197	0.00036
32	0.00484	0.00594	0.00391	0.02742	0.00226	0.00165	0.00353	0.00054	0.00109
33	0.0	0.0	0.0	0.0	0.0	0.00004	0.00017	0.00014	0.00248
34	0.00000	0.00001	0.00000	0.0	0.00000	0.00000	0.00001	0.00000	0.00001
35	0.0	0.00163	0.00483	0.00064	0.00965	0.0	0.00229	0.00063	0.00000
36	0.00058	0.07793	0.03248	0.00278	0.00003	0.00000	0.00004	0.0	0.00000
37	0.02566	0.04246	0.01622	0.01107	0.00001	0.0	0.00038	0.00037	0.00005
38	0.00500	0.01658	0.01668	0.05381	0.00057	0.00115	0.00026	0.00012	0.0
39	0.0	0.0	0.0	0.0	0.02425	0.00128	0.0	0.0	0.0
40	0.00028	0.09906	0.05223	0.0	0.0	0.0	0.0	0.0	0.0
41	0.00114	0.00168	0.00124	0.00735	0.00282	0.00006	0.00011	0.00006	0.0
42	0.00104	0.01658	0.00309	0.01072	0.00137	0.00150	0.00083	0.00037	0.00120
43	0.00049	0.00004	0.00002	0.00129	0.0	0.0	0.0	0.0	0.0
44	0.0	0.00005	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.04134	0.00325	0.00126	0.0	0.0	0.0	0.0	0.0	0.0
46	0.00764	0.00477	0.00049	0.0	0.0	0.0	0.0	0.0	0.0
47	0.0	0.00002	0.00001	0.01197	0.00022	0.00006	0.00021	0.00019	0.00000
48	0.0	0.0	0.0	0.0	0.0	0.0	0.00601	0.00045	0.0
49	0.00225	0.00521	0.00111	0.00625	0.00003	0.00003	0.00011	0.00012	0.00001
50	0.00015	0.00005	0.00002	0.09704	0.00001	0.00002	0.00002	0.0	0.0
51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.00374	0.00129	0.0	0.0	0.0	0.0	0.0	0.0
53	0.00751	0.00809	0.00467	0.02176	0.00013	0.00001	0.00006	0.00006	0.0
54	0.0	0.00392	0.00354	0.0	0.0	0.0	0.0	0.0	0.0
55	0.0	0.01517	0.00729	0.01312	0.00036	0.00010	0.00027	0.00021	0.0
56	0.0	0.00070	0.00128	0.03394	0.0	0.0	0.0	0.0	0.0
57	0.0	0.00004	0.00001	0.0	0.0	0.0	0.0	0.0	0.0
58	0.00015	0.00028	0.00029	0.0	0.00006	0.00001	0.00001	0.0	0.00000
59	0.00360	0.00002	0.00001	0.0	0.0	0.0	0.0	0.0	0.0
60	0.0	0.0	0.0	0.15220	0.0	0.0	0.0	0.0	0.0
61	0.00057	0.00005	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.00021	0.00365	0.00094	0.02007	0.00000	0.0	0.00000	0.00001	0.00000
63	0.00008	0.00000	0.00000	0.0	0.0	0.0	0.0	0.0	0.0
64	0.00024	0.00160	0.00287	0.00175	0.00049	0.00111	0.00101	0.01005	0.01826
65	0.03231	0.03448	0.01766	0.01237	0.04147	0.01209	0.02514	0.02440	0.00834
66	0.00078	0.00207	0.00107	0.00491	0.00247	0.00038	0.00158	0.00299	0.00300
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.05615	0.00286	0.00148	0.00387	0.00555	0.00094	0.01147	0.00772	0.00309
69	0.02399	0.09467	0.08170	0.03202	0.03744	0.01305	0.02990	0.04753	0.03545
70	0.00511	0.00830	0.00292	0.00775	0.00528	0.00200	0.00599	0.00817	0.00657
71	0.01211	0.00400	0.00207	0.00414	0.00420	0.00123	0.00429	0.00816	0.01124
72	0.00105	0.0	0.0	0.00101	0.00059	0.00042	0.00166	0.00134	0.00258
73	0.00828	0.04721	0.00347	0.01153	0.01509	0.03006	0.00515	0.00460	0.00649
74	0.0	0.0	0.0	0.0	0.00008	0.0	0.00019	0.0	0.0
75	0.0	0.00503	0.00133	0.0	0.00447	0.00030	0.00055	0.00059	0.00020
76	0.00003	0.0	0.0	0.00003	0.00001	0.00001	0.00003	0.00003	0.00005
77	0.00108	0.00111	0.00057	0.00113	0.00103	0.00100	0.00109	0.00107	0.00116
78	0.00083	0.0	0.0	0.00080	0.00044	0.00211	0.00074	0.00142	0.00180
79	0.00024	0.00025	0.00013	0.00015	0.00047	0.00006	0.00023	0.00024	0.00009

COL. / ROW	19	20	21	22	23	24	25	26	27
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00118
3	0.00038	0.10344	0.0	0.0	0.0	0.0	0.0	0.0	0.00146
4	0.0	0.00110	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00480
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00628
7	0.0	0.00018	0.0	0.00068	0.0	0.00806	0.00039	0.0	0.00603
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.01949
9	0.0	0.0	0.0	0.0	0.0	0.00393	0.0	0.0	0.00189
10	0.0	0.00000	0.0	0.0	0.0	0.00159	0.0	0.0	0.02052
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00015	0.00183	0.00009	0.00051	0.00019	0.00431	0.00348	0.00326	0.00049
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.00007	0.0	0.0	0.00921	0.0	0.00835	0.0	0.0	0.00709
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.37359	0.0	0.0	0.06088	0.00072	0.00544	0.0	0.0	0.00004
17	0.08656	0.00013	0.0	0.01165	0.01716	0.00086	0.0	0.00133	0.00002
18	0.00045	0.00146	0.0	0.00033	0.00116	0.00075	0.00093	0.0	0.00051
19	0.07772	0.00013	0.0	0.00037	0.00007	0.0	0.0	0.0	0.00290
20	0.0	0.31667	0.33362	0.11860	0.05934	0.07008	0.0	0.00003	0.00290
21	0.0	0.0	0.04318	0.00008	0.0	0.00059	0.0	0.0	0.00029
22	0.0	0.0	0.0	0.01530	0.00325	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.02209	0.0	0.0	0.0	0.0
24	0.00349	0.00747	0.00193	0.00370	0.00256	0.21701	0.44377	0.17096	0.00710
25	0.01153	0.00464	0.0	0.01945	0.02107	0.02501	0.03221	0.00330	0.00628
26	0.00379	0.00503	0.00227	0.01001	0.00199	0.00829	0.00254	0.15539	0.00732
27	0.00009	0.00777	0.0	0.00005	0.00011	0.03610	0.00186	0.01467	0.22399
28	0.0	0.00753	0.0	0.00026	0.00105	0.00987	0.00279	0.0	0.00148
29	0.00017	0.00117	0.00058	0.00002	0.00001	0.00230	0.00158	0.00041	0.00054
30	0.00013	0.00505	0.00076	0.02165	0.01880	0.00016	0.0	0.0	0.00102
31	0.00079	0.01028	0.00657	0.00213	0.00208	0.01321	0.00700	0.00072	0.01232
32	0.03664	0.00685	0.00057	0.04146	0.00817	0.01439	0.00667	0.00097	0.00605
33	0.0	0.0	0.0	0.00115	0.00152	0.0	0.0	0.0	0.0
34	0.0	0.0	0.00001	0.0	0.0	0.0	0.0	0.00001	0.00001
35	0.0	0.00117	0.0	0.01445	0.04325	0.0	0.0	0.0	0.00130
36	0.0	0.00398	0.00150	0.00256	0.00152	0.00118	0.00003	0.0	0.00299
37	0.0	0.00004	0.04565	0.02323	0.10461	0.0	0.00032	0.0	0.00941
38	0.0	0.00104	0.0	0.00859	0.01332	0.00112	0.0	0.00107	0.00989
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00800
40	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
41	0.00018	0.00150	0.00218	0.00365	0.00360	0.00138	0.00100	0.00016	0.00023
42	0.00126	0.00832	0.00651	0.06230	0.03899	0.01111	0.00224	0.00076	0.00431
43	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
46	0.0	0.00042	0.0	0.0	0.0	0.0	0.0	0.0	0.00054
47	0.00018	0.00012	0.00011	0.00089	0.00261	0.00081	0.00053	0.00009	0.00014
48	0.0	0.00161	0.00186	0.00120	0.00037	0.00282	0.00362	0.00270	0.01135
49	0.00014	0.00135	0.00018	0.00013	0.00011	0.00059	0.00051	0.00008	0.00015
50	0.00005	0.00034	0.00033	0.00008	0.00003	0.00039	0.00027	0.00004	0.00031
51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
53	0.0	0.00003	0.0	0.00005	0.00007	0.00045	0.00009	0.00002	0.00130
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
55	0.0	0.00132	0.00227	0.00039	0.00031	0.00136	0.00043	0.00008	0.00011
56	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
57	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
58	0.0	0.00010	0.00009	0.00002	0.00000	0.00001	0.00001	0.00001	0.00002
59	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
60	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.0	0.00073	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00017
63	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00404	0.00004
64	0.01595	0.00035	0.00046	0.00072	0.00017	0.00040	0.00052	0.00189	0.00018
65	0.01074	0.05115	0.03964	0.02302	0.01913	0.04020	0.03428	0.01504	0.03660
66	0.00279	0.00284	0.00223	0.00552	0.00434	0.00314	0.00215	0.01314	0.00527
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.00360	0.00716	0.00856	0.00594	0.00562	0.02055	0.00489	0.00428	0.02915
69	0.05774	0.04534	0.05471	0.05795	0.06075	0.04274	0.03817	0.02408	0.03204
70	0.00513	0.00696	0.00647	0.00542	0.00549	0.00662	0.00642	0.00978	0.01070
71	0.01269	0.00687	0.01015	0.01261	0.01137	0.00407	0.00970	0.03753	0.01149
72	0.00189	0.00207	0.00288	0.00222	0.00180	0.00107	0.00133	0.00140	0.00072
73	0.00445	0.00502	0.00517	0.00783	0.00883	0.00954	0.00462	0.02128	0.00849
74	0.0	0.0	0.0	0.0	0.0	0.00021	0.0	0.0	0.00246
75	0.00144	0.00884	0.00885	0.00167	0.00198	0.00080	0.00091	0.00103	0.00152
76	0.00003	0.00004	0.00005	0.00004	0.00003	0.00002	0.00003	0.00004	0.00001
77	0.00112	0.00112	0.00119	0.00114	0.00112	0.00105	0.00109	0.00113	0.00104
78	0.00164	0.00039	0.00098	0.00057	0.00102	0.00111	0.00088	0.00725	0.00108
79	0.00033	0.00073	0.00056	0.00019	0.00017	0.00138	0.00009	0.00015	0.00062

COL. ROW	28	29	30	31	32	33	34	35	36
1	0.0	0.00072	0.0	0.0	0.0	0.06064	0.0	0.0	0.0
2	0.0	0.00037	0.0	0.0	0.0	0.0	0.0	0.0	0.00053
3	0.0	0.0	0.0	0.0	0.0	0.00107	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.00005	0.0	0.0	0.0	0.0	0.00123
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00004	0.00051
7	0.00603	0.00049	0.00007	0.00059	0.00145	0.00251	0.0	0.00134	0.00482
8	0.0	0.0	0.0	0.53071	0.0	0.0	0.0	0.0	0.0
9	0.0	0.00056	0.00026	0.00284	0.00059	0.0	0.0	0.01167	0.06264
10	0.0	0.0	0.0	0.0	0.00111	0.00093	0.0	0.00029	0.00137
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00709	0.00025	0.00039	0.00134	0.00076	0.00008	0.00011	0.00035	0.00030
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.00429	0.02299	0.03269	0.00054	0.00005	0.24101	0.00023	0.0	0.00060
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.00076	0.0	0.00012	0.0	0.02156	0.0	0.02506	0.0	0.00161
17	0.0	0.0	0.0	0.0	0.06945	0.0	0.01346	0.00002	0.00014
18	0.00050	0.00037	0.00042	0.00016	0.00111	0.0	0.00185	0.00125	0.0
19	0.0	0.0	0.0	0.0	0.00013	0.00156	0.0	0.0	0.00028
20	0.00017	0.0	0.0	0.0	0.00054	0.0	0.00822	0.01411	0.00108
21	0.00023	0.0	0.0	0.0	0.0	0.00215	0.00032	0.00554	0.00155
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.05446	0.00889	0.00685	0.00311	0.00232	0.00306	0.00841	0.00189	0.02185
25	0.00594	0.03351	0.01130	0.00164	0.00700	0.00142	0.01239	0.07903	0.00996
26	0.00582	0.06331	0.00768	0.01174	0.01782	0.00182	0.01064	0.00651	0.00843
27	0.31083	0.09889	0.22647	0.02710	0.04539	0.07227	0.00033	0.03982	0.02032
28	0.02918	0.00010	0.13237	0.0	0.14222	0.0	0.00067	0.0	0.01140
29	0.00977	0.06983	0.00676	0.00216	0.00064	0.02619	0.00032	0.00154	0.00635
30	0.0	0.00132	0.00167	0.0	0.00003	0.00002	0.0	0.00002	0.00050
31	0.00474	0.00351	0.00478	0.07274	0.00214	0.00412	0.00025	0.00317	0.00831
32	0.00208	0.00992	0.00297	0.00039	0.03292	0.00476	0.05848	0.00385	0.00540
33	0.0	0.0	0.0	0.0	0.0	0.14492	0.21500	0.0	0.0
34	0.00000	0.0	0.00001	0.00000	0.0	0.0	0.08590	0.00000	0.00000
35	0.0	0.02208	0.0	0.00004	0.00673	0.0	0.00004	0.05014	0.00005
36	0.0	0.00191	0.00446	0.00130	0.00140	0.00849	0.00239	0.02758	0.13059
37	0.0	0.0	0.00585	0.0	0.00124	0.0	0.0	0.0	0.00332
38	0.0	0.0	0.00028	0.0	0.00031	0.00019	0.00010	0.00213	0.00151
39	0.00099	0.01749	0.04984	0.00698	0.0	0.0	0.0	0.0	0.0
40	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
41	0.00088	0.00353	0.00038	0.00010	0.00060	0.00066	0.00050	0.00423	0.00085
42	0.00105	0.00870	0.00129	0.01114	0.01088	0.00153	0.00766	0.00395	0.01235
43	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
46	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
47	0.00157	0.00011	0.00007	0.00015	0.00047	0.00071	0.00006	0.00149	0.00047
48	0.00096	0.00029	0.0	0.0	0.0	0.0	0.0	0.0	0.00013
49	0.00063	0.00002	0.00017	0.00022	0.00039	0.00013	0.00008	0.00133	0.00014
50	0.00026	0.0	0.00008	0.00001	0.00274	0.00012	0.00001	0.00069	0.00018
51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
53	0.00069	0.00003	0.0	0.00014	0.00026	0.00059	0.00000	0.00050	0.00078
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
55	0.00043	0.00003	0.00016	0.00002	0.00109	0.00094	0.00017	0.00072	0.00370
56	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
57	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
58	0.00001	0.00000	0.00002	0.00002	0.0	0.00001	0.00000	0.00000	0.00004
59	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
60	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.00020	0.00085	0.00012	0.00005	0.00034	0.00032	0.0	0.00045	0.00035
63	0.00005	0.00004	0.00004	0.00002	0.00010	0.00009	0.0	0.00009	0.00008
64	0.00069	0.00046	0.00076	0.00003	0.0	0.00057	0.00097	0.00042	0.00055
65	0.03463	0.01973	0.03040	0.04931	0.02270	0.02357	0.01215	0.02420	0.06354
66	0.00284	0.00195	0.00413	0.00120	0.00300	0.00302	0.00297	0.00305	0.00345
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.00943	0.00326	0.00364	0.01434	0.01062	0.00793	0.00328	0.03803	0.02946
69	0.02087	0.02795	0.04453	0.01008	0.03588	0.02648	0.02752	0.03659	0.03045
70	0.00746	0.00804	0.00746	0.00632	0.00568	0.00609	0.00748	0.00976	0.01005
71	0.00586	0.00789	0.01182	0.00771	0.01041	0.00298	0.00910	0.00733	0.00735
72	0.00072	0.00049	0.00056	0.0	0.00132	0.00130	0.00298	0.00179	0.00147
73	0.00826	0.10780	0.00910	0.00852	0.00909	0.00447	0.02178	0.01073	0.00770
74	0.00320	0.0	0.0	0.00033	0.0	0.0	0.0	0.00093	0.0
75	0.00033	0.00039	0.00152	0.00105	0.00034	0.00065	0.00033	0.00073	0.00342
76	0.00002	0.0	0.00001	0.0	0.00002	0.00003	0.00006	0.00004	0.00002
77	0.00104	0.00103	0.00104	0.00097	0.00108	0.00107	0.00119	0.00111	0.00110
78	0.00547	0.00223	0.00226	0.00183	0.00075	0.00527	0.00230	0.00265	0.00083
79	0.00034	0.00022	0.00022	0.00038	0.00028	0.00038	0.00008	0.00042	0.00037

COL. ROW	37	38	39	40	41	42	43	44	45
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00150	0.0
5	0.05700	0.00195	0.0	0.0	0.0	0.00438	0.0	0.0	0.0
6	0.00017	0.09157	0.0	0.0	0.0	0.00106	0.0	0.0	0.0
7	0.02760	0.00120	0.00018	0.00021	0.00036	0.00237	0.00119	0.00090	0.00065
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.00308	0.00048	0.0	0.00011	0.0	0.00014	0.0	0.00013	0.00009
10	0.00037	0.00002	0.0	0.0	0.0	0.00004	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00679	0.00026	0.00019	0.00082	0.00055	0.00046	0.00030	0.00058	0.00025
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.00034	0.00001	0.0	0.0	0.0	0.0	0.0	0.0	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.00176	0.0	0.0	0.0	0.00068	0.00005	0.00007	0.00002
17	0.00004	0.00052	0.0	0.0	0.0	0.00090	0.00019	0.0	0.0
18	0.00079	0.00056	0.0	0.00098	0.00121	0.00125	0.00075	0.00078	0.00080
19	0.00009	0.00019	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.00063	0.00102	0.00034	0.00115	0.00437	0.00796	0.00020	0.00374	0.00112
21	0.00016	0.0	0.00048	0.00102	0.00051	0.00039	0.0	0.00009	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00064	0.0
24	0.00259	0.00304	0.00195	0.00173	0.00479	0.00100	0.00083	0.00046	0.00084
25	0.00057	0.00059	0.01352	0.00411	0.09634	0.00602	0.00571	0.00227	0.00067
26	0.00403	0.00466	0.00773	0.00593	0.03656	0.00866	0.00798	0.02176	0.00834
27	0.00798	0.00910	0.0	0.00238	0.00275	0.01161	0.00005	0.00085	0.00072
28	0.00023	0.01318	0.00097	0.00020	0.00379	0.00068	0.00051	0.00016	0.00049
29	0.00200	0.00113	0.00241	0.00026	0.00057	0.00007	0.00052	0.00063	0.00055
30	0.00075	0.00078	0.01724	0.00277	0.00581	0.00173	0.00114	0.00439	0.00150
31	0.00802	0.00463	0.00267	0.00580	0.00735	0.00526	0.00446	0.00412	0.00470
32	0.00316	0.00134	0.01506	0.00106	0.00573	0.00701	0.00503	0.03849	0.01643
33	0.0	0.0	0.00039	0.00012	0.00004	0.00006	0.00016	0.00122	0.00012
34	0.00000	0.0	0.00000	0.0	0.0	0.0	0.00001	0.0	0.00001
35	0.00006	0.00003	0.00001	0.00406	0.00084	0.00017	0.00040	0.00011	0.00003
36	0.01622	0.00474	0.00197	0.00590	0.00850	0.00884	0.00731	0.00573	0.00579
37	0.22163	0.00184	0.45558	0.25908	0.21365	0.15594	0.11500	0.15882	0.16971
38	0.01247	0.35426	0.00827	0.07808	0.07393	0.06416	0.03498	0.00628	0.00481
39	0.0	0.0	0.00159	0.0	0.0	0.0	0.0	0.0	0.0
40	0.0	0.0	0.0	0.02101	0.0	0.0	0.0	0.0	0.01031
41	0.00598	0.00979	0.00935	0.01574	0.03231	0.01509	0.02211	0.03293	0.01010
42	0.01630	0.01003	0.00312	0.02827	0.01857	0.04317	0.00153	0.00410	0.00987
43	0.0	0.0	0.0	0.00009	0.00140	0.0	0.10848	0.05478	0.02690
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.04196	0.00053
45	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.06617
46	0.0	0.0	0.0	0.0	0.0	0.0	0.00123	0.0	0.0
47	0.00598	0.00701	0.01266	0.00472	0.00667	0.02706	0.01884	0.01739	0.01480
48	0.00069	0.0	0.0	0.00001	0.00023	0.00010	0.0	0.0	0.0
49	0.00246	0.00270	0.00707	0.00840	0.0	0.00100	0.02399	0.06336	0.05691
50	0.00823	0.00370	0.00501	0.00342	0.00331	0.00104	0.03851	0.01625	0.00406
51	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
52	0.0	0.0	0.0	0.00138	0.0	0.0	0.0	0.0	0.0
53	0.00428	0.00091	0.00103	0.00975	0.00061	0.00335	0.00813	0.00480	0.01025
54	0.0	0.0	0.0	0.0	0.00019	0.0	0.0	0.00205	0.0
55	0.00012	0.00453	0.00040	0.00088	0.00171	0.00067	0.00036	0.00062	0.00043
56	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
57	0.0	0.0	0.0	0.0	0.00028	0.00041	0.0	0.0	0.0
58	0.00001	0.00001	0.00001	0.00003	0.00001	0.00002	0.02076	0.00803	0.00108
59	0.0	0.0	0.0	0.0	0.0	0.0	0.00141	0.00451	0.00036
60	0.0	0.0	0.0	0.0	0.00022	0.0	0.0	0.0	0.0
61	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.00027	0.00021	0.00018	0.00888	0.00004	0.00033	0.00075	0.00120	0.00005
63	0.00008	0.00005	0.00006	0.00009	0.00011	0.00010	0.00005	0.00007	0.00006
64	0.00063	0.00078	0.00075	0.00069	0.00015	0.00051	0.00160	0.00055	0.00056
65	0.05382	0.02283	0.02597	0.01953	0.01729	0.01845	0.01388	0.01712	0.01548
66	0.00364	0.00301	0.00097	0.00425	0.00253	0.00292	0.00246	0.00246	0.00321
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.02420	0.02465	0.00674	0.00639	0.01005	0.00982	0.00421	0.00548	0.00640
69	0.03670	0.03552	0.03968	0.03957	0.03216	0.04026	0.02942	0.04411	0.03302
70	0.00762	0.00668	0.00624	0.00860	0.00885	0.00769	0.00656	0.00669	0.00736
71	0.00379	0.00407	0.00400	0.00592	0.00883	0.00555	0.00436	0.00517	0.00508
72	0.00114	0.00081	0.00086	0.00133	0.00165	0.00172	0.00105	0.00111	0.00109
73	0.00627	0.00480	0.00477	0.00584	0.00606	0.00856	0.00635	0.00822	0.00633
74	0.00092	0.00055	0.0	0.00027	0.0	0.0	0.00679	0.0	0.0
75	0.00031	0.00062	0.00038	0.00223	0.00080	0.00109	0.00028	0.00106	0.00084
76	0.00003	0.00002	0.00002	0.00002	0.00003	0.00003	0.00002	0.00002	0.00002
77	0.00107	0.00104	0.00104	0.00109	0.00112	0.00112	0.00107	0.00107	0.00108
78	0.00080	0.00048	0.00069	0.00103	0.00112	0.00111	0.00098	0.00210	0.00072
79	0.00075	0.00032	0.00009	0.00023	0.00030	0.00027	0.00015	0.00034	0.00015

COL. ROW	46	47	48	49	50	51	52	53	54
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00070	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.0	0.0	0.00001	0.00019	0.00008	0.0	0.00039	0.00034	0.00033
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.00307	0.0	0.0	0.0	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00036	0.00199	0.00083	0.00114	0.00426	0.00073	0.00113	0.00129	0.00051
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.00186	0.0	0.00128	0.00036	0.0	0.0	0.00003	0.00019	0.00277
17	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00032	0.00008
18	0.00086	0.00130	0.00117	0.00108	0.00146	0.00091	0.00081	0.00098	0.00073
19	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20	0.00014	0.00148	0.00527	0.00132	0.0	0.00028	0.00259	0.00133	0.00133
21	0.0	0.0	0.00009	0.0	0.0	0.00037	0.00625	0.00003	0.00623
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.00052	0.0	0.00165	0.00309	0.00003	0.00555	0.00472	0.00878	0.00127
25	0.00014	0.00023	0.00288	0.00162	0.0	0.00123	0.01013	0.00393	0.01018
26	0.00862	0.00404	0.04028	0.00471	0.00317	0.01865	0.00348	0.00401	0.07588
27	0.0	0.00072	0.00156	0.00146	0.0	0.00025	0.00788	0.00343	0.00525
28	0.00053	0.00022	0.00063	0.00004	0.0	0.00104	0.00117	0.00773	0.00268
29	0.00088	0.00045	0.00059	0.00030	0.00063	0.0	0.00020	0.00010	0.00015
30	0.00318	0.00005	0.00027	0.00060	0.0	0.00102	0.00609	0.00326	0.00619
31	0.00384	0.00592	0.00653	0.00412	0.01147	0.00128	0.00323	0.00342	0.00102
32	0.02475	0.00029	0.00915	0.00381	0.00036	0.01343	0.01364	0.00700	0.03677
33	0.00007	0.00006	0.00149	0.00026	0.00035	0.00006	0.00014	0.00011	0.00024
34	0.00001	0.00001	0.00001	0.00000	0.00000	0.0	0.0	0.00000	0.00142
35	0.00023	0.00012	0.00002	0.00001	0.00002	0.00025	0.00152	0.00084	0.00116
36	0.00427	0.00791	0.00392	0.00894	0.01753	0.00242	0.00704	0.00862	0.00843
37	0.13114	0.07751	0.09862	0.11476	0.08010	0.02574	0.07406	0.06256	0.07720
38	0.01152	0.02132	0.04758	0.02573	0.08125	0.02485	0.06121	0.07023	0.04322
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
40	0.00894	0.00097	0.00890	0.00673	0.0	0.0	0.00254	0.00240	0.00236
41	0.01545	0.00862	0.01114	0.01112	0.00169	0.01514	0.04355	0.01515	0.05308
42	0.01158	0.00374	0.01306	0.01437	0.01244	0.00559	0.02631	0.00637	0.03402
43	0.00960	0.0	0.00009	0.00239	0.0	0.0	0.00133	0.00150	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.00028	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
46	0.04737	0.00097	0.00002	0.0	0.0	0.0	0.0	0.0	0.0
47	0.01362	0.07102	0.01257	0.01678	0.01592	0.01284	0.00177	0.01107	0.00792
48	0.0	0.00041	0.05991	0.0	0.0	0.0	0.0	0.0	0.0
49	0.05930	0.03264	0.05128	0.08557	0.00183	0.00687	0.00411	0.00794	0.00962
50	0.01925	0.00135	0.00314	0.00338	0.07688	0.00211	0.00092	0.00142	0.00091
51	0.00013	0.0	0.00047	0.0	0.0	0.10949	0.0	0.0	0.0
52	0.0	0.0	0.0	0.0	0.0	0.0	0.06059	0.0	0.01886
53	0.06094	0.02822	0.04034	0.04276	0.00435	0.02889	0.11125	0.07782	0.03634
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.01231
55	0.00141	0.00014	0.00020	0.00058	0.00065	0.00460	0.00810	0.00867	0.00791
56	0.0	0.0	0.0	0.0	0.0	0.00000	0.0	0.0	0.0
57	0.0	0.0	0.0	0.00014	0.0	0.05467	0.00006	0.02014	0.0
58	0.00294	0.00001	0.00002	0.00017	0.00002	0.0	0.00001	0.0	0.00000
59	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
60	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
62	0.0	0.00045	0.00115	0.00482	0.00067	0.00040	0.01451	0.00085	0.03298
63	0.00007	0.00014	0.00006	0.00014	0.00009	0.00007	0.00007	0.00008	0.00005
64	0.00055	0.00060	0.00051	0.00022	0.00026	0.00010	0.00141	0.00035	0.00051
65	0.01426	0.00869	0.01227	0.01528	0.01047	0.00880	0.01787	0.01291	0.01670
66	0.00362	0.00983	0.01144	0.01083	0.00748	0.00689	0.00363	0.00432	0.00431
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.00409	0.00689	0.00586	0.00671	0.00851	0.00435	0.00533	0.00793	0.00619
69	0.05246	0.03344	0.04431	0.05438	0.03393	0.06459	0.06708	0.03157	0.04387
70	0.00924	0.00884	0.00775	0.00681	0.00881	0.00538	0.00929	0.00463	0.00276
71	0.01068	0.02097	0.01055	0.00745	0.01616	0.01006	0.01503	0.00782	0.00446
72	0.00124	0.00187	0.00167	0.00142	0.00210	0.00129	0.00110	0.00135	0.00102
73	0.01003	0.00778	0.00843	0.00846	0.00735	0.01205	0.00622	0.00823	0.02207
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.00063	0.00075	0.00162	0.00073	0.00131	0.00025	0.00116	0.00049	0.00018
76	0.00002	0.00004	0.00003	0.00003	0.00004	0.00002	0.00001	0.00003	0.00002
77	0.00106	0.00113	0.00113	0.00111	0.00116	0.00108	0.00109	0.00110	0.00105
78	0.00145	0.00084	0.00104	0.00110	0.00127	0.00152	0.00113	0.00274	0.00225
79	0.00008	0.00012	0.00020	0.00017	0.00019	0.0	0.00018	0.00018	0.00019

COL. ROW	55	56	57	58	59	60	61	62	63
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00161	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.00148	0.0	0.0	0.0	0.0	0.0
6	0.00096	0.0	0.0	0.0	0.0	0.0	0.0	0.00049	0.0
7	0.0	0.00033	0.00017	0.0	0.00063	0.00019	0.00062	0.0	0.00169
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.00019	0.0	0.0	0.00005	0.0	0.0
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00006	0.00141	0.00055	0.00007	0.00312	0.00178	0.00023	0.00017	0.00034
13	0.0	0.0	0.0	0.0	0.0	0.01702	0.0	0.0	0.0
14	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00408	0.0
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.0	0.00016	0.0	0.00023	0.00233	0.00014	0.00056	0.00885	0.00031
17	0.0	0.0	0.0	0.0	0.00409	0.00042	0.00070	0.00177	0.00022
18	0.00104	0.00081	0.00152	0.00108	0.00044	0.00098	0.00114	0.00099	0.00092
19	0.0	0.0	0.0	0.0	0.00676	0.0	0.00042	0.0	0.0
20	0.00047	0.00188	0.00011	0.0	0.00039	0.00168	0.02531	0.00031	0.00007
21	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00034	0.0
22	0.0	0.02611	0.00544	0.0	0.0	0.0	0.00700	0.0	0.0
23	0.0	0.0	0.0	0.0	0.00019	0.00138	0.00405	0.0	0.0
24	0.00273	0.00459	0.00537	0.00090	0.00378	0.00034	0.00156	0.00450	0.03506
25	0.01964	0.00562	0.00611	0.00806	0.00097	0.00068	0.00035	0.00914	0.00711
26	0.00652	0.01733	0.00201	0.01436	0.01221	0.00208	0.00361	0.00585	0.03919
27	0.00533	0.00091	0.01332	0.02589	0.00177	0.00118	0.00229	0.00238	0.06501
28	0.02094	0.00773	0.00858	0.00583	0.00116	0.00060	0.00796	0.00235	0.00055
29	0.00016	0.00037	0.00005	0.00001	0.00084	0.00043	0.00064	0.0	0.0
30	0.00528	0.00039	0.00062	0.00001	0.00404	0.00086	0.00872	0.00050	0.00017
31	0.00171	0.00112	0.00167	0.00116	0.00204	0.00254	0.00456	0.00124	0.00262
32	0.01611	0.01076	0.00695	0.05584	0.02858	0.00687	0.00913	0.01310	0.00750
33	0.0	0.00018	0.0	0.00006	0.00029	0.0	0.00044	0.00049	0.00007
34	0.0	0.0	0.0	0.00000	0.00000	0.00001	0.0	0.00059	0.00034
35	0.03258	0.00451	0.03714	0.00035	0.01074	0.00013	0.00549	0.00336	0.01153
36	0.00592	0.00332	0.00734	0.01114	0.00260	0.00380	0.00878	0.00392	0.00300
37	0.06598	0.00968	0.01897	0.03116	0.08852	0.03316	0.12037	0.02042	0.00515
38	0.04677	0.01980	0.04996	0.10905	0.01003	0.02968	0.01668	0.04657	0.02948
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00293	0.0
40	0.0	0.00045	0.00008	0.0	0.00029	0.00006	0.04476	0.00078	0.0
41	0.02815	0.02064	0.02648	0.02745	0.03179	0.02108	0.00413	0.02157	0.00597
42	0.01647	0.01321	0.01482	0.00281	0.03789	0.01021	0.02086	0.01311	0.01078
43	0.0	0.0	0.0	0.0	0.00242	0.00015	0.02898	0.0	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.0	0.0	0.0	0.0	0.0	0.0	0.00495	0.0	0.0
46	0.0	0.0	0.0	0.0	0.0	0.0	0.00352	0.0	0.0
47	0.00515	0.00434	0.00628	0.01577	0.01106	0.02034	0.00472	0.01384	0.00452
48	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
49	0.00009	0.00027	0.00040	0.02016	0.00418	0.01024	0.01530	0.00430	0.0
50	0.00161	0.00084	0.00111	0.00200	0.00607	0.01006	0.00347	0.00579	0.0
51	0.0	0.0	0.0	0.0	0.0	0.00006	0.0	0.00767	0.0
52	0.0	0.00065	0.0	0.0	0.00116	0.00006	0.00255	0.00011	0.0
53	0.01581	0.00931	0.00298	0.00855	0.00083	0.00341	0.03539	0.02276	0.00896
54	0.0	0.0	0.0	0.0	0.0	0.0	0.00563	0.0	0.0
55	0.04581	0.01188	0.00428	0.02016	0.00415	0.00171	0.00338	0.00331	0.00267
56	0.0	0.06324	0.00020	0.0	0.00510	0.02631	0.00153	0.00461	0.0
57	0.0	0.19299	0.06572	0.00776	0.00067	0.00579	0.00009	0.03182	0.0
58	0.02141	0.0	0.0	0.04498	0.01483	0.00403	0.00112	0.00005	0.0
59	0.0	0.0	0.0	0.00429	0.31162	0.00350	0.00711	0.0	0.0
60	0.0	0.0	0.0	0.0	0.0	0.20568	0.0	0.0	0.0
61	0.0	0.0	0.0	0.0	0.00006	0.0	0.07429	0.0	0.0
62	0.00023	0.00165	0.00082	0.00064	0.00423	0.01369	0.00130	0.07324	0.00054
63	0.00004	0.00005	0.00014	0.00007	0.00004	0.00228	0.00009	0.00081	0.06098
64	0.00333	0.00030	0.00096	0.00048	0.00087	0.00060	0.00218	0.00482	0.00040
65	0.01405	0.01251	0.01160	0.01429	0.01917	0.00917	0.01816	0.01096	0.01534
66	0.00264	0.00349	0.00302	0.00345	0.00199	0.00488	0.00270	0.00514	0.00294
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.00508	0.00280	0.00835	0.00580	0.00472	0.00587	0.00646	0.00395	0.00318
69	0.06994	0.03780	0.06257	0.03820	0.03045	0.01860	0.05040	0.04522	0.03854
70	0.00433	0.00347	0.00514	0.00397	0.00371	0.00284	0.00506	0.00558	0.00529
71	0.00973	0.00561	0.02308	0.00713	0.00243	0.00566	0.00433	0.01110	0.01280
72	0.00142	0.00110	0.00218	0.00142	0.00064	0.0	0.00163	0.00155	0.00127
73	0.00712	0.00919	0.00898	0.00933	0.01422	0.00213	0.00516	0.00980	0.02313
74	0.0	0.00036	0.00087	0.0	0.00053	0.00106	0.0	0.0	0.00138
75	0.00024	0.00022	0.00002	0.00047	0.00034	0.00008	0.00092	0.00025	0.00055
76	0.00003	0.00002	0.00005	0.00001	0.00001	0.0	0.00003	0.00003	0.00001
77	0.00109	0.00109	0.00115	0.00107	0.00103	0.00096	0.00109	0.00114	0.00110
78	0.00150	0.00263	0.00371	0.00259	0.00175	0.00099	0.00074	0.00120	0.00215
79	0.00019	0.00010	0.00020	0.00024	0.00018	0.00017	0.00023	0.00215	0.00008

COL. ROW	64	65	66	67	68	69	70	71	72
1	0.0	0.00001	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.00164	0.00023	0.0	0.0	0.0	0.0	0.0	0.00037	0.0
3	0.00037	0.00004	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.00169	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.00002	0.00082	0.0	0.0	0.03154	0.0	0.00023	0.0	0.0
8	0.0	0.0	0.0	0.0	0.05569	0.0	0.0	0.0	0.0
9	0.00006	0.00004	0.0	0.0	0.0	0.0	0.0	0.00008	0.0
10	0.00001	0.00003	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00301	0.04195	0.03163	0.00479	0.06044	0.00755	0.00191	0.10504	0.00125
13	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.00146	0.00305	0.0	0.0	0.0	0.00003	0.0	0.0	0.00112
15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.02043	0.00020	0.00008	0.0	0.0	0.0	0.0	0.0	0.01068
17	0.00663	0.00048	0.00035	0.00113	0.0	0.00018	0.00083	0.0	0.00195
18	0.00070	0.00013	0.0	0.0	0.0	0.0	0.00001	0.0	0.00819
19	0.00030	0.00045	0.00050	0.00162	0.0	0.00023	0.00117	0.0	0.01061
20	0.01751	0.00027	0.00002	0.0	0.0	0.00004	0.0	0.0	0.00041
21	0.00037	0.00052	0.0	0.0	0.0	0.00096	0.0	0.0	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00090
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00027
24	0.02256	0.00102	0.0	0.00539	0.00048	0.00471	0.00396	0.0	0.01228
25	0.04376	0.00025	0.0	0.00128	0.0	0.00377	0.00094	0.0	0.00159
26	0.01543	0.00488	0.01539	0.01539	0.00251	0.01275	0.02689	0.00412	0.00887
27	0.00690	0.00110	0.00003	0.0	0.00001	0.0	0.0	0.00032	0.00832
28	0.02501	0.00000	0.0	0.0	0.0	0.0	0.0	0.0	0.0
29	0.00093	0.00036	0.00003	0.00003	0.00002	0.0	0.00043	0.0	0.01492
30	0.00841	0.00121	0.0	0.0	0.0	0.0	0.0	0.0	0.00001
31	0.00299	0.04729	0.00153	0.00127	0.01268	0.00749	0.00354	0.00518	0.01208
32	0.02417	0.00794	0.00053	0.00055	0.00044	0.00170	0.00206	0.00062	0.00579
33	0.00684	0.00010	0.0	0.0	0.0	0.0	0.0	0.0	0.0
34	0.00545	0.00001	0.00003	0.00003	0.00001	0.00004	0.00004	0.00000	0.00057
35	0.00538	0.00016	0.0	0.0	0.0	0.00109	0.0	0.0	0.00026
36	0.00166	0.00012	0.0	0.0	0.00117	0.00046	0.0	0.00000	0.00440
37	0.02741	0.00118	0.0	0.0	0.00010	0.0	0.0	0.0	0.0
38	0.05164	0.00150	0.00254	0.0	0.0	0.0	0.0	0.0	0.00053
39	0.0	0.0	0.0	0.0	0.0	0.00004	0.0	0.0	0.0
40	0.0	0.00002	0.0	0.0	0.0	0.0	0.0	0.0	0.0
41	0.00870	0.00047	0.0	0.0	0.0	0.00021	0.0	0.0	0.0
42	0.01644	0.00128	0.00032	0.0	0.00910	0.00010	0.0	0.0	0.00238
43	0.0	0.00249	0.0	0.0	0.0	0.0	0.0	0.0	0.0
44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
45	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
46	0.0	0.00039	0.0	0.0	0.0	0.0	0.0	0.0	0.0
47	0.00031	0.00072	0.00001	0.0	0.00011	0.0	0.0	0.0	0.0
48	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00005
49	0.00042	0.00044	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	0.00217	0.00022	0.0	0.0	0.0	0.00014	0.0	0.0	0.0
51	0.0	0.00008	0.0	0.0	0.00042	0.00008	0.00019	0.0	0.0
52	0.00010	0.00006	0.0	0.0	0.0	0.00005	0.0	0.0	0.00326
53	0.00576	0.00094	0.00005	0.0	0.00048	0.0	0.0	0.0	0.00056
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.00990
55	0.00297	0.00022	0.0	0.0	0.00013	0.00004	0.0	0.0	0.00063
56	0.0	0.00053	0.01272	0.02250	0.00018	0.00003	0.0	0.0	0.00034
57	0.00148	0.00074	0.00058	0.0	0.0	0.0	0.0	0.0	0.01929
58	0.00002	0.00221	0.00048	0.0	0.00002	0.00024	0.00007	0.00002	0.00030
59	0.0	0.00273	0.0	0.0	0.00000	0.00178	0.0	0.0	0.0
60	0.0	0.00511	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.0	0.00907	0.00048	0.0	0.00006	0.0	0.00036	0.0	0.00074
62	0.00012	0.00078	0.00001	0.0	0.0	0.0	0.0	0.0	0.00588
63	0.0	0.00000	0.00000	0.0	0.0	0.0	0.0	0.0	0.00865
64	0.06252	0.00148	0.00083	0.00589	0.00031	0.00048	0.00088	0.0	0.02206
65	0.01645	0.06459	0.00157	0.00166	0.02163	0.00387	0.00916	0.00048	0.00740
66	0.00539	0.00842	0.00858	0.03992	0.00272	0.01107	0.01599	0.00320	0.00524
67	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
68	0.00433	0.00449	0.00574	0.00405	0.18967	0.02094	0.00477	0.00411	0.01972
69	0.06605	0.03082	0.00444	0.01186	0.01313	0.01731	0.00989	0.00303	0.04374
70	0.00824	0.02123	0.00503	0.00795	0.00614	0.01696	0.20915	0.02436	0.01604
71	0.01385	0.03159	0.01471	0.05056	0.00321	0.05511	0.07969	0.02541	0.04689
72	0.00200	0.0	0.0	0.0	0.0	0.00244	0.0	0.0	0.02999
73	0.01114	0.01269	0.01348	0.03398	0.01075	0.00398	0.02802	0.00953	0.01771
74	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
75	0.00137	0.02504	0.00140	0.00109	0.00130	0.00900	0.00321	0.00083	0.00968
76	0.00003	0.00076	0.00033	0.19594	0.0	0.00104	0.00019	0.0	0.0
77	0.00117	0.00096	0.00096	0.00097	0.00083	0.00103	0.00546	0.00023	0.00099
78	0.00104	0.00128	0.00263	0.00055	0.00258	0.00448	0.01256	0.00408	0.00034
79	0.00024	0.00171	0.00040	0.00009	0.00032	0.00055	0.00028	0.00004	0.00157

COL. ROW	73	74	75	76	77	78	79
1	0.0	0.0	0.0	0.00198	0.00020	0.00055	0.00234
2	0.0	0.0	0.0	0.0	0.00024	0.19768	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.00057	0.0	0.0	0.00006
5	0.0	0.0	0.0	0.0	0.0	0.00053	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	0.00185	0.0	0.00134	0.0	0.00001	0.01076	0.0
8	0.0	0.0	0.0	0.0	0.0	0.0	0.0
9	0.0	0.0	0.0	0.0	0.0	0.00011	0.0
10	0.0	0.0	0.0	0.0	0.0	0.0	0.0
11	0.0	0.0	0.0	0.0	0.0	0.0	0.0
12	0.00126	0.0	0.01270	0.02142	0.02915	0.00050	0.35223
13	0.0	0.0	0.0	0.0	0.0	0.00032	0.0
14	0.0	0.0	0.0	0.0	0.00754	0.08238	0.0
15	0.0	0.0	0.0	0.0	0.00001	0.0	0.0
16	0.0	0.0	0.0	0.0	0.00011	0.0	0.0
17	0.00087	0.00033	0.00207	0.00058	0.00100	0.0	0.00226
18	0.00027	0.00008	0.00015	0.0	0.00169	0.0	0.00213
19	0.00130	0.00048	0.00226	0.00095	0.00147	0.00057	0.0
20	0.00019	0.0	0.0	0.0	0.00014	0.0	0.0
21	0.0	0.0	0.0	0.0	0.0	0.0	0.0
22	0.0	0.0	0.0	0.0	0.0	0.0	0.0
23	0.0	0.0	0.0	0.0	0.0	0.0	0.0
24	0.00428	0.00127	0.00042	0.00059	0.00374	0.00761	0.00014
25	0.00050	0.00030	0.00012	0.00014	0.00089	0.00132	0.0
26	0.00529	0.00009	0.00434	0.02164	0.01690	0.00963	0.01059
27	0.00085	0.0	0.00001	0.0	0.00018	0.0	0.03068
28	0.00027	0.0	0.0	0.0	0.0	0.0	0.00005
29	0.00172	0.00105	0.00119	0.00016	0.02591	0.0	0.00332
30	0.0	0.0	0.00777	0.0	0.0	0.0	0.00002
31	0.00662	0.00006	0.00347	0.00056	0.00311	0.00080	0.00064
32	0.00424	0.00321	0.03487	0.00028	0.00285	0.00024	0.0
33	0.0	0.0	0.0	0.0	0.0	0.0	0.0
34	0.00003	0.0	0.00001	0.00175	0.00011	0.00023	0.0
35	0.0	0.0	0.01225	0.0	0.00027	0.0	0.0
36	0.00001	0.0	0.00504	0.0	0.00000	0.00347	0.0
37	0.00029	0.0	0.0	0.0	0.0	0.0	0.00187
38	0.00150	0.0	0.0	0.0	0.0	0.0	0.0
39	0.0	0.0	0.0	0.0	0.0	0.0	0.0
40	0.0	0.0	0.0	0.0	0.0	0.0	0.0
41	0.0	0.0	0.0	0.0	0.00090	0.0	0.0
42	0.00022	0.0	0.01464	0.0	0.00001	0.0	0.0
43	0.00638	0.0	0.0	0.0	0.0	0.0	0.0
44	0.00842	0.0	0.0	0.0	0.0	0.0	0.0
45	0.00108	0.0	0.0	0.0	0.0	0.0	0.0
46	0.0	0.0	0.0	0.0	0.0	0.0	0.0
47	0.00112	0.00017	0.00012	0.0	0.0	0.0	0.0
48	0.0	0.0	0.0	0.0	0.0	0.0	0.0
49	0.0	0.0	0.0	0.0	0.0	0.0	0.0
50	0.00003	0.0	0.01376	0.0	0.00000	0.00021	0.00074
51	0.00517	0.0	0.0	0.0	0.00013	0.0	0.0
52	0.00523	0.00028	0.0	0.0	0.0	0.0	0.0
53	0.00001	0.0	0.00078	0.0	0.0	0.0	0.0
54	0.0	0.0	0.0	0.0	0.0	0.0	0.0
55	0.00001	0.0	0.00426	0.0	0.00000	0.00004	0.0
56	0.0	0.0	0.0	0.0	0.00046	0.0	0.0
57	0.0	0.0	0.0	0.0	0.0	0.0	0.0
58	0.00028	0.0	0.01488	0.0	0.00051	0.00011	0.0
59	0.00031	0.0	0.14430	0.0	0.00007	0.00262	0.00920
60	0.0	0.0	0.0	0.0	0.0	0.0	0.0
61	0.00111	0.00107	0.00119	0.00016	0.00085	0.0	0.0
62	0.0	0.0	0.00203	0.0	0.01140	0.0	0.0
63	0.01168	0.0	0.0	0.00432	0.00313	0.0	0.0
64	0.00460	0.00092	0.00023	0.01589	0.00143	0.0	0.00007
65	0.00708	0.00038	0.00949	0.00405	0.00529	0.22369	0.01478
66	0.01684	0.00037	0.00706	0.00550	0.00843	0.00108	0.00874
67	0.00042	0.0	0.0	0.0	0.0	0.0	0.0
68	0.01484	0.0	0.01939	0.00755	0.01852	0.00500	0.00464
69	0.02612	0.00218	0.08582	0.01263	0.01852	0.01431	0.00868
70	0.01723	0.00094	0.02630	0.02300	0.01167	0.0	0.0
71	0.05480	0.00218	0.03956	0.05117	0.06899	0.00180	0.01234
72	0.0	0.0	0.0	0.0	0.00487	0.0	0.00895
73	0.03191	0.00375	0.01658	0.02019	0.02270	0.00873	0.02037
74	0.0	0.0	0.0	0.0	0.00173	0.00040	0.0
75	0.00725	0.0	0.01703	0.0	0.00215	0.00940	0.00100
76	0.0	0.00094	0.0	0.24182	0.00370	0.0	0.0
77	0.00022	0.00010	0.00097	0.00098	0.01306	0.0	0.00001
78	0.04062	0.0	0.00042	0.00045	0.00070	0.00084	0.00087
79	0.00045	0.0	0.00008	0.00022	0.00054	0.00012	0.00010

TABLE F-7

LIST OF SELECTED CONTROL TOTALS AND PARAMETERS, UNITED STATES ECONOMY, 1958

(amounts are given in thousands of 1958 dollars)

COL. 1 - TOTAL INDUSTRY OUTPUT LEVELS (DOMESTIC)
COL. 2 - TOTAL SECONDARY PRODUCTS TRANSFER-OUT
COL. 3 - PRIMARY PRODUCT OUTPUT (DOMESTIC) OF EACH INDUSTRY : (Col. 1 minus Col. 2)
COL. 4 - PRIMARY PRODUCT SPECIALIZATION RATIO FOR EACH INDUSTRY : (Col. 3 divided by Col. 1)
COL. 5 - TOTAL SECONDARY PRODUCTS TRANSFERS-IN
COL. 6 - TOTAL DOMESTIC PRIMARY PRODUCTS : (Col. 3 plus Col. 5)
COL. 7 - PRIMARY PRODUCTS OF EACH INDUSTRY AS A PROPORTION OF TOTAL DOMESTIC SUPPLY OF PRIMARY PRODUCTS, WHEREVER PRODUCED
COL. 8 - TOTAL INTERMEDIATE DEMAND FOR DOMESTICALLY PRODUCED PRODUCTS AS A PROPORTION OF TOTAL DOMESTIC SUPPLY OF PRIMARY PRODUCTS, WHEREVER PRODUCED
COL. 9 - COMPETITIVE IMPORTS VECTOR
COL. 10 - SAME AS COL. 9 EXCEPT ALL ELEMENTS ARE POSITIVE
COL. 11 - SAME AS COL. 10 EXCEPT ELEMENTS 65,69,AND 70 ARE OMITTED

COL. / ROW	1	2	3	4	5	6
1	26241197	2276320	23964877	.913254	6438	23971315
2	23043440	2374017	20669423	.896976	0	20669423
3	922383	16897	905486	.981681	226537	1132023
4	1018326	5555	1012771	.994545	545207	1557978
5	776716	30236	746480	.961072	18896	765376
6	1016674	28780	987894	.971692	14011	1001905
7	2748109	18195	2729914	.993379	435	2730349
8	9668889	535075	9133814	.944660	0	9133814
9	1428361	76369	1351992	.946534	103198	1455190
10	468553	160123	308430	.658261	14912	323342
11	52416000	0	52416000	1.000000	0	52416000
12	16867670	0	16867670	1.000000	0	16867670
13	4148456	1847439	2301017	.554668	507792	2808809
14	63299560	2679077	60620483	.957676	1517323	62137806
15	5915445	148342	5767103	.974923	300	5767403
16	10400164	467334	9932830	.955065	151500	10084330
17	2026130	173105	1853025	.914564	151000	2004025
18	14191962	159743	14032219	.988744	61500	14093719
19	1852247	122013	1730234	.934127	426101	2156335
20	7644186	215113	7429073	.971859	283472	7712545
21	408882	31915	376967	.921946	36000	412967
22	3177285	124262	3053023	.960891	122500	3175523
23	1351155	105062	1246093	.922243	143630	1389723
24	9296833	595810	8701023	.935913	228400	8929423
25	3530286	94121	3436165	.973339	83250	3519415
26	12397234	6431870	5965364	.481185	163500	6128864
27	10337068	988596	9348472	.904364	1433190	10781662
28	3765800	359359	3406441	.904573	466750	3873191
29	6219967	440771	5779196	.929136	372287	6151483
30	1815079	112602	1702477	.937963	59100	1761577
31	16860221	846324	16013897	.949804	472305	16486202
32	6535540	472896	6062644	.927642	301713	6364357
33	873360	11180	862180	.987199	7446	869626
34	3046701	87153	2959548	.971394	55500	3015048
35	2122914	35399	2087515	.983325	21618	2109133
36	7297716	352483	6945233	.951700	209963	7155196
37	19152726	999943	18152783	.947791	322815	18475598
38	8808295	471795	8336500	.946437	300957	8637457
39	2049819	44086	2005733	.978493	34500	2040233
40	7338592	575022	6763570	.921644	664300	7427870
41	3265724	365983	2899741	.887932	371278	3271019
42	5483217	505023	4978194	.907897	930949	5909143
43	1951054	303144	1647910	.844626	245500	1893410
44	2303688	256511	2047177	.888652	126252	2173429
45	2854969	331535	2523434	.883874	219159	2742593
46	902629	125163	777466	.861335	175241	952707
47	3070300	323865	2746435	.894517	548187	3294622
48	2223521	261887	1961634	.882220	255000	2216634
49	3235802	367747	2868055	.886351	490159	3358214
50	1437754	110847	1326907	.922903	113500	1440407
51	2118441	564770	1553671	.733403	84878	1638549
52	1931998	271092	1660906	.859683	305841	1966747
53	4670739	687184	3983555	.852875	424499	4408054
54	3421897	518987	2902910	.848334	172495	3075405
55	2151483	268124	1883359	.875377	135160	2018519
56	5548468	827769	4720699	.850811	372813	5093512
57	2393748	316772	2076976	.867667	254946	2331922
58	1369915	193535	1176380	.858725	157000	1333380
59	22425176	1039591	21385585	.953642	276467	21662052
60	11922621	2342365	9580256	.803536	742357	10322613
61	3596965	274940	3322025	.923563	122000	3444025
62	3025349	466884	2558465	.845676	431732	2990197
63	1461198	167689	1293509	.885239	80219	1373728
64	5004935	781115	4223820	.843931	274935	4498755
65	32501290	2998328	29502962	.907747	721214	30224176
66	9292088	405518	8886570	.956359	0	8886570
67	1548537	1531935	16602	.010721	0	16602
68	17211165	58658	17152507	.996592	3055853	20208360
69	92203247	1258132	90945115	.986355	3047238	93992353
70	26401032	768567	25632465	.970889	63000	25695465
71	55274311	0	55274311	1.000000	6659984	61934295
72	12169295	1320617	10848678	.891480	1856	10850534
73	16447810	568584	15879226	.965431	7992844	23872070
74	534270	0	534270	1.000000	4805034	5339304
75	7891796	69794	7822002	.991156	21000	7843002
76	5619749	248901	5370848	.955710	0	5370848
77	22703262	599699	22103563	.973585	0	22103563
78	4105041	960776	3144265	.765952	0	3144265
79	4783900	4042000	741900	.155083	0	741900

COL. / ROW	7	8	9	10	11
1	.999731	.884959	230943	230943	230943
2	1.000000	.720827	351336	351336	351336
3	.799883	.828494	302265	302265	302265
4	.650055	.999850	0	0	0
5	.975311	.976410	449235	449235	449235
6	.986016	.837029	288097	288097	288097
7	.999841	.768813	3000	3000	3000
8	1.000000	1.001352	1183500	1183500	1183500
9	.929083	.955112	116344	116344	116344
10	.953882	.759168	79511	79511	79511
11	1.000000	0.000000	0	0	0
12	1.000000	.737970	0	0	0
13	.819214	.098285	12334	12334	12334
14	.975581	.233497	1300424	1300424	1300424
15	.999948	.191922	26517	26517	26517
16	.984977	.912958	253546	253546	253546
17	.924652	.594695	323089	323089	323089
18	.995636	.197264	33039	33039	33039
19	.802396	.433579	8160	8160	8160
20	.963245	.958160	511492	511492	511492
21	.912826	1.010175	5526	5526	5526
22	.961424	.170472	0	0	0
23	.896648	.209230	0	0	0
24	.974422	.867336	982359	982359	982359
25	.976346	.982712	3986	3986	3986
26	.973323	.541011	39834	39834	39834
27	.867072	.828690	336768	336768	336768
28	.879492	.919598	39736	39736	39736
29	.939480	.284525	43721	43721	43721
30	.966451	.975081	1657	1657	1657
31	.971351	.464057	654420	654420	654420
32	.952593	.727799	33314	33314	33314
33	.991438	.970693	36928	36928	36928
34	.981592	.103011	15035	15035	15035
35	.989750	.907659	61261	61261	61261
36	.970656	.951044	101398	101398	101398
37	.982527	.972474	284100	284100	284100
38	.965157	.926216	907166	907166	907166
39	.983090	.967275	408	408	408
40	.910567	.873974	19943	19943	19943
41	.886495	.906096	27187	27187	27187
42	.842456	.846231	106175	106175	106175
43	.870340	.421335	6564	6564	6564
44	.941911	.141580	125862	125862	125862
45	.920091	.250366	708	708	708
46	.816060	.380880	14705	14705	14705
47	.833612	.526472	37402	37402	37402
48	.884961	.183263	32020	32020	32020
49	.854042	.568770	9285	9285	9285
50	.921203	.943568	17088	17088	17088
51	.948199	.170196	43086	43086	43086
52	.844494	.294434	4944	4944	4944
53	.903699	.553883	56623	56623	56623
54	.943911	.144859	1100	1100	1100
55	.933040	.802342	11037	11037	11037
56	.926806	.222512	41004	41004	41004
57	.890671	.806029	1659	1659	1659
58	.882254	.614510	21253	21253	21253
59	.987237	.357857	633710	633710	633710
60	.928084	.304269	58477	58477	58477
61	.964576	.181248	58359	58359	58359
62	.855618	.434470	113371	113371	113371
63	.941605	.360748	76466	76466	76466
64	.938886	.293413	247557	247557	247557
65	.976138	.554546	17460	17460	0
66	1.000000	.471792	0	0	0
67	1.000000	.457897	0	0	0
68	.848783	.558165	35870	35870	35870
69	.967580	.280500	631300	631300	0
70	.997548	.531882	45311	45311	0
71	.892467	.325547	0	0	0
72	.999829	.097933	0	0	0
73	.665180	.866639	0	0	0
74	.100064	.030398	0	0	0
75	.997322	.413669	0	0	0
76	1.000000	.345325	0	0	0
77	1.000000	.055355	0	0	0
78	1.000000	.740304	0	0	0
79	1.000000	.414728	0	0	0

TABLE F-8

THE H MATRIX AND THE $\hat{g}$ MATRIX, UNITED STATES ECONOMY, 1958

(for detailed explanations, refer to Chapter IV, Section B)

ROW / COL	The H Matrix: 65	The H Matrix: 69	The H Matrix: 70	The $\hat{g}$ Matrix Diagonal Elements: Row and Col.	$\hat{g}_i$
1	.001516	.054820	.003935	1	.009634
2	.001516	.054820	.003935	2	.016998
3	.001516	.054820	.003935	3	.267013
4	.001516	.054820	.003935	4	0.000000
5	.001516	.054820	.003935	5	.586947
6	.001516	.054820	.003935	6	.287549
7	.001516	.054820	.003935	7	.001099
8	.001516	.054820	.003935	8	.129573
9	.001516	.054820	.003935	9	.079951
10	.001516	.054820	.003935	10	.245904
11	.001516	.054820	.003935	11	0.000000
12	.001516	.054820	.003935	12	0.000000
13	.001516	.054820	.003935	13	.004391
14	.001516	.054820	.003935	14	.020928
15	.001516	.054820	.003935	15	.004598
16	.001516	.054820	.003935	16	.025143
17	.001516	.054820	.003935	17	.161220
18	.001516	.054820	.003935	18	.002344
19	.001516	.054820	.003935	19	.003784
20	.001516	.054820	.003935	20	.066319
21	.001516	.054820	.003935	21	.013381
22	.001516	.054820	.003935	22	0.000000
23	.001516	.054820	.003935	23	0.000000
24	.001516	.054820	.003935	24	.110014
25	.001516	.054820	.003935	25	.001133
26	.001516	.054820	.003935	26	.006499
27	.001516	.054820	.003935	27	.031235
28	.001516	.054820	.003935	28	.010259
29	.001516	.054820	.003935	29	.007107
30	.001516	.054820	.003935	30	.000941
31	.001516	.054820	.003935	31	.039695
32	.001516	.054820	.003935	32	.005234
33	.001516	.054820	.003935	33	.042464
34	.001516	.054820	.003935	34	.004987
35	.001516	.054820	.003935	35	.029046
36	.001516	.054820	.003935	36	.014171
37	.001516	.054820	.003935	37	.015377
38	.001516	.054820	.003935	38	.105027
39	.001516	.054820	.003935	39	.000200
40	.001516	.054820	.003935	40	.002685
41	.001516	.054820	.003935	41	.008311
42	.001516	.054820	.003935	42	.017968
43	.001516	.054820	.003935	43	.003467
44	.001516	.054820	.003935	44	.057909
45	.001516	.054820	.003935	45	.000258
46	.001516	.054820	.003935	46	.015435
47	.001516	.054820	.003935	47	.011352
48	.001516	.054820	.003935	48	.014445
49	.001516	.054820	.003935	49	.002765
50	.001516	.054820	.003935	50	.011863
51	.001516	.054820	.003935	51	.026295
52	.001516	.054820	.003935	52	.002514
53	.001516	.054820	.003935	53	.012845
54	.001516	.054820	.003935	54	.000358
55	.001516	.054820	.003935	55	.005468
56	.001516	.054820	.003935	56	.008050
57	.001516	.054820	.003935	57	.000711
58	.001516	.054820	.003935	58	.015939
59	.001516	.054820	.003935	59	.029254
60	.001516	.054820	.003935	60	.005665
61	.001516	.054820	.003935	61	.016945
62	.001516	.054820	.003935	62	.037914
63	.001516	.054820	.003935	63	.055663
64	.001516	.054820	.003935	64	.055028
65	0.000000	0.000000	0.000000	65	0.000000
66	.001516	.054820	.003935	66	0.000000
67	.001516	.054820	.003935	67	0.000000
68	.001516	.054820	.003935	68	.001775
69	0.000000	0.000000	0.000000	69	0.000000
70	0.000000	0.000000	0.000000	70	0.000000
71	.001516	.054820	.003935	71	0.000000
72	.001516	.054820	.003935	72	0.000000
73	.001516	.054820	.003935	73	0.000000
74	.001516	.054820	.003935	74	0.000000
75	.001516	.054820	.003935	75	0.000000
76	.001516	.054820	.003935	76	0.000000
77	.001516	.054820	.003935	77	0.000000
78	.001516	.054820	.003935	78	0.000000
79	.001516	.054820	.003935	79	0.000000

TABLE F-9

EXOGENOUS INFORMATION FOR 1961 USED IN THE EXPERIMENTS
(in millions of constant 1958 dollars)

Sector No.	Final Demand Vector	Final Demand Vector Minus Competitive Imports	Total Domestic Product Output	Competitive Imports Required for Intermediate Consumption
1	2678.7	2473.9	25125.1	204.8
2	4910.0	4659.7	21274.5	250.3
3	202.2	- 118.1	1172.7	320.3
4	- 32.0	- 32.0	1595.4	0.0
5	63.2	- 389.3	939.3	452.5
6	264.4	34.6	1243.1	229.8
7	432.9	431.5	2527.9	1.4
8	86.2	-1068.7	55.4	1154.9
9	30.8	- 82.7	1620.0	113.5
10	112.3	13.4	378.2	98.9
11	55336.8	55336.8	55336.8	0.0
12	4902.9	4902.9	17913.3	0.0
13	2056.7	2039.4	2344.5	17.3
14	52336.0	51156.9	67151.4	1179.1
15	5382.4	5275.4	6429.8	107.0
16	1166.4	864.3	11243.5	302.1
17	883.0	450.5	2337.0	432.5
18	12562.7	12502.7	15493.7	60.0
19	1358.0	1346.9	2423.5	11.1
20	215.4	- 399.5	8167.6	614.9
21	3.4	- 2.2	432.4	5.6
22	2713.4	2713.4	3239.0	0.0
23	1290.1	1290.1	1558.6	0.0
24	1600.9	503.3	10367.4	1097.6
25	84.0	81.3	4144.6	2.7
26	3422.3	3366.2	7260.7	56.1
27	2293.1	1846.2	12960.5	446.9
28	646.5	605.7	5126.5	40.8
29	5405.2	5340.4	7476.0	64.8
30	64.2	62.9	1965.2	1.3
31	9734.6	9025.5	17900.0	709.1
32	2154.4	2087.9	8183.5	66.5
33	44.4	1.4	855.0	43.0
34	2715.0	2693.0	3012.7	22.0
35	205.4	128.6	2488.2	76.8
36	389.1	252.9	7845.7	136.2
37	878.1	334.4	20019.8	543.7
38	766.6	192.6	10167.5	959.2
39	74.4	74.1	2277.8	0.3
40	920.0	897.6	7729.6	22.4
41	379.2	334.9	3488.3	44.3
42	1051.0	896.7	6501.5	154.3
43	1023.3	1003.5	1725.0	19.8
44	1607.1	1497.5	1961.6	109.6
45	2030.2	2024.2	2629.0	6.0
46	609.3	591.5	1051.5	17.8
47	1872.9	1831.3	3650.9	41.6
48	2341.7	2264.3	2782.7	77.4
49	1527.9	1499.0	3783.6	28.9
50	82.1	48.1	1737.9	34.0
51	1902.5	1795.0	2243.8	107.5
52	1750.2	1736.7	2485.0	13.5
53	2404.0	2335.5	5328.9	68.5
54	3009.8	3009.2	3542.4	0.6
55	505.1	484.5	2332.4	20.6
56	6655.8	6455.7	8018.5	200.1
57	623.3	605.0	3263.7	18.3
58	603.7	571.2	1601.4	32.5
59	17281.9	16851.9	27333.0	430.0
60	7921.2	7788.0	10649.8	133.2
61	2719.5	2649.4	3391.8	70.1
62	2314.5	2191.6	3825.6	122.9
63	1043.6	934.0	1556.8	109.6
64	3565.4	3224.1	5143.3	341.3
65	13336.5	13330.7	32707.9	5.8
66	5734.5	5734.5	10477.7	0.0
67	14.9	14.9	22.6	0.0
68	10429.3	10377.9	24201.9	51.4
69	75265.9	75265.9	104443.7	0.0
70	13566.9	13566.9	27723.8	0.0
71	47374.6	47374.6	70346.1	0.0
72	11058.7	11058.7	12076.7	0.0
73	3682.9	3682.9	27134.6	0.0
74	6396.3	6396.2	6584.1	.1
75	4908.3	4908.3	8545.8	0.0
76	3871.5	3871.5	5817.4	0.0
77	23188.0	23188.0	24552.0	0.0
78	876.0	876.0	3435.5	0.0
79	537.0	533.8	846.7	3.2

SELECTED BIBLIOGRAPHY

A. MAJOR BIBLIOGRAPHICAL REFERENCES

Riley, Vera, and Allen, R. L. *Interindustry Economic Studies: A Comprehensive Bibliography on Interindustry Research.* Baltimore: Johns Hopkins University Press, 1955.

Taskier, Charlotte E. *Input-Output Bibliography, 1955-1960.* New York: United Nations, 1961.

United Nations, Statistical Office, *Input-Output Bibliography, 1960-1963.* New York: United Nations, 1964.

United Nations, Statistical Office. *Input-Output Bibliography, 1963-1966.* New York: United Nations, Statistical Office, Statistical Papers, Series M., No. 46, 1967.

B. BOOKS

Alexandroff, P., and Hopf, H. *Topologie I.* Berlin: Springer-Verlag, 1935.

Almon, Clopper, Jr. *The American Economy to 1975.* New York: Harper and Row, Publishers, 1966.

————. *Matrix Methods in Economics.* Reading, Mass.: Addison-Wesley Publishing Co., 1967.

Arrow, Kenneth J., and Hoffenberg, Marvin. *A Time Series Analysis of Interindustry Demands.* Amsterdam: North-Holland Publishing Co., 1959.

Arrow, Kenneth J., Karlin, Samuel, and Suppes, Patrick (eds.). *Mathematical Methods in the Social Sciences.* Stanford, Calif.: Stanford University Press, 1960.

Artle, Roland. *The Structure of the Stockholm Economy, Toward a Framework for Projecting Metropolitan Community Development.* Ithaca, N. Y.: Cornell University Press, 1965.

Barna, Tibor (ed.). *The Structural Interdependence of the Economy.* The Report of an International Conference Held in Italy in 1954. New York: John Wiley and Sons, Inc., 1955.

————. *Structural Interdependence and Economic Development.* Proceedings of an International Conference on Input-Output Techniques, Geneva, September 1961. London: MacMillan and Co., Ltd. and New York: St. Martin's Press, 1963.

Baumol, William J. *Economic Theory and Operations Analysis.* Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1961.

Bellman, Richard. *Introduction to Matrix Analysis.* New York: McGraw-Hill Book Company, Inc., 1960.

Bodewig, E. *Matrix Calculus.* Revised Edition. New York: Interscience Publishers, Inc., 1959.

Bruno, Michael. *Interdependence, Resource Use and Structural Change in Israel.* Jerusalem: Bank of Israel, 1961.

Cambridge, Department of Applied Economics. *Input-Output Relationships, 1954-1960, No. 3 in "A Programme for Growth."* London: Chapman and Hall, 1963 and Cambridge, Mass.: The M.I.T. Press, 1963.

Chenery, Hollis B., and Clark, Paul G. *Interindustry Economics.* New York: John Wiley and Sons, Inc., February 1965.

Chenery, H. B., Clark, P. G., and Cao-Pinna, V. *The Structure and Growth of the Italian Economy.* Rome: U. S. Mutual Security Agency, 1953.

Chipman, J. S. *The Theory of Inter-Sectoral Money Flows and Income Formation.* Baltimore: The John Hopkins Press, 1951.

Clark, Paul G., and Chenery, Hollis B. *Interindustry Economics.* New York: John Wiley and Sons, Inc., Fourth Printing, February, 1965.

Dantzig, George B. *Linear Programming and Extensions.* Princeton, New Jersey: Princeton University Press, 1963.

Dorfman, Robert, Samuelson, Paul A., and Solow, Robert M. *Linear Programming and Economic Analysis.* New York: McGraw-Hill Book Co., Inc., 1958.

Dorn, William S., and Greenberg, Herbert J. *Mathematics and Computing: With FORTRAN Programming.* New York: John Wiley and Sons, Inc., 1967.

Duesenberry, J. S., *et al.* (eds.). *The Brookings-SSRC Quarterly Econometric Model of the United States.* Chicago: Rand McNally and Co. and Amsterdam: North-Holland Publishing Co., 1966.

Edwards, J. W. (ed.). *Reissner Anniversary Volume.* Ann Arbor, Michigan: University of Michigan Press, 1949.

Faddeev, D. K., and Faddeeva, V. N. *Computational Methods of Linear Algebra.* Trans. Robert C. Williams. San Francisco: W. H. Freeman and Co., 1963.

Fritz, John. *Lectures on Advanced Numerical Analysis.* New York: Gordon and Breach, Science Publishers, Inc., 1967.

Gantmacher, F. R. *Applications of the Theory of Matrices,* translated and revised by J. L. Brenner, D. W. Bushaw, and S. Evanusa. New York: Interscience Publishers, Inc., 1959.

Geary, R. C. (ed.). *Europe's Future in Figures.* Amsterdam: North-Holland Publishing Co., 1962.

Ghosh, A. *Experiments with Input-Output Models.* Cambridge, England: The University Press, 1964.

Green, H. A. John. *Aggregation in Economic Analysis: An Introductory Survey.* Princeton, N. J.: Princeton University Press, 1964.

Hadamard, J. *Lecons sur la propagation des ondes.* Paris: Herman, 1903.

Harmston, Floyd K., and Lund, Richard E. *Application of an Input-Output Framework to a Community Economic System.* Columbia, Missouri: University of Missouri Press, 1967.

Hatanaka, Michio. *The Workability of Input-Output Analysis.* Ludwigshafen am Rhein: Fachverlag fur Wirtschaftstheorie und Ökonometrie, 1960.

Hochwald, Werner (ed.). *Design of Regional Accounts.* Baltimore: The Johns Hopkins Press, 1961.

Hochwald, Werner, Striver, Herbert E., and Sonenblum, Sydney. *Local Impact of Foreign Trade.* Washington, D. C.: National Planning Association, 1960.

Höglund, Bengt, and Werin, Lars. *The Production System of the Swedish Economy, An Input-Output Study.* Stockholm: Almqvist and Wiksell, 1964.

Householder, A. S. *The Theory of Matrices in Numerical Analysis.* New York: Blaisdell Publishing Co., A Division of Ginn and Co., First Edition, 1964.

Isard, Walter, and Cumberland, J. E. (eds.). *Regional Economic Planning.* Paris: European Productivity Agency of O. E. E. C., 1961 (Conference on Regional Economic Planning, Bellagio, June, 1960).

James, M. L., Smith, G. M., and Wolford, J. C. *Applied Numerical Methods for Digital Computation with FORTRAN.* Scranton, Penn.: International Textbook Co., 1967.

Karlin, Samuel. *A First Course in Stochastic Processes.* New York: Academic Press, 1966.

Kelly, Louis G. *Handbook of Numerical Methods and Applications.* Reading, Mass.: Addison-Wesley Publishing Co., 1967.

Koopmans, T. C. (ed.). *Activity Analysis of Production and Allocation.* New York: John Wiley and Sons, Inc., 1951.

Koutsoyiannis, A., and Ganas, A. *Input-Output Table of the Greek Economy (Year 1960).* Athens: Center of Planning and Economic Research, 1967.

Leontief, Wassily. *Input-Output Economics.* New York: Oxford University Press, 1966.

————. *The Structure of the American Economy.* New York: Oxford University Press, Second Edition, 1951.

————. *The Structure of the American Economy, 1919-1939.* New York: Oxford University Press, Second Edition, 1951.

Leontief, Wassily, *et al. Studies in the Structure of the American Economy.* New York: Oxford University Press, Second Edition, 1951.

Lukács, O., *et al.* (eds.). *Input-Output Tables - Their Compilation and Use.* Scientific Conference on Statistical Problems, Budapest, June 1-5, 1961. Budapest: Branch A. Akadémiai Kiadó (Publishing House of the Hungarian Academy of Sciences), 1962.

Manne, A. S., and Markowitz, H. *Studies in Process Analysis.* New York: McGraw-Hill Book Co., Inc., 1963.

Margulis, B. E. *Systems of Linear Equations.* Translated and adapted from the Russian by Jerome Kristian and Daniel A. Levine. New York: The Macmillan Co., 1964.

Miernyk, William H. *The Elements of Input-Output Analysis.* New York: Random House, Inc., 1965.

Morgenstern, Oskar (ed.). *Economic Activity Analysis.* New York: John Wiley and Sons, Inc., 1954.

Mosak, S. L. *General Equilibrium Theory in International Trade.* Bloomington, Indiana: Principia Press, 1944.

National Bureau of Economic Research. *Input-Output Analysis: An Appraisal.* Studies in Income and Wealth, Vol. XVIII. Princeton, N. J.: Princeton University Press, 1955.

Noble, Ben. *Applied Linear Algebra.* Preliminary Edition. Englewood-Cliffs, N. J.: Prentice-Hall, Inc., 1966.

Norkin, S. B. *The Elements of Computational Mathematics.* Trans. G. J. Tee and English translation edited by A. D. Booth. New York: The Macmillan Co., 1965.

Nugent, Jeffrey B. *Programming the Optimal Development of the Greek Economy, 1954-1961.* Center of Planning and Economic Research, Research Monograph Series, No. 15. Athens: Constantinidis and Mihalas, 1966.

Quesnay, Francois. *Tableau Economique.* London: British Economic Association, 1894.

Rall, Louis B. (ed.). *Error in Digital Computation.* I. New York: John Wiley and Sons, Inc., 1965.

Ralston, Anthony. *A First Course in Numerical Analysis.* New York: McGraw-Hill Book Co., Inc., 1965.

Rasmussen, P. Norregaard. *Studies in Inter-Sectoral Relations.* Copenhagen: Einer Harcks, 1956.

Sandee, J. *A Long-Term Planning Model for India.* New York: United Nations, 1959.

Schneider, Hans (ed.). *Recent Advances in Matrix Theory.* Madison and Milwaukee, Wisc.: The University of Wisconsin Press, 1964.

Schwartz, Jacob. *Lectures on the Mathematical Method in Analytical Economics.* New York: Gordon and Breach, Science Publishers, 1961.

Searle, S. R. *Matrix Algebra for the Biological Sciences (Including Applications in Statistics).* New York: John Wiley and Sons, Inc., 1966).

Skolka, J. *The Aggregation Problem in Input-Output Analysis.* Prague: Czechoslovakian Academy of Sciences, 1964.

Stone, Richard. *Input-Output and National Accounts.* Paris: Organization for European Economic Cooperation, 1961.

Stone, Richard. *Mathematics in the Social Sciences and Other Essays.* London: Chapman and Hall, Ltd., 1966.

Szego, G. (ed.). *Studies in Mathematical Analysis and Related Topics.* Stanford, Calif.: Stanford University Press, 1962.

Theil, Henri. *Applied Economic Forecasting.* Amsterdam: North-Holland Publishing Co., and Chicago: Rand McNally and Co., 1966.

————. *Economics and Information Theory.* Amsterdam: North-Holland Publishing Co., 1967.

Tilanus, C. B. *Input-Output Experiments, the Netherlands, 1948-1961.* Rotterdam: Rotterdam University Press, 1966.

Varga, Richard S. *Matrix Iterative Analysis.* Englewood Cliffs, N. J.: Prentice-Hall, Inc., Third Printing, 1965.

Vilenkin, N. Ya. *Successive Approximation.* Translated and adapted from the Russian by M. B. P. Slater and J. W. Teller. New York: The Macmillan Co., 1964.

Walras, Leon. *Elements of Pure Economics.* Trans. William Jaffe. Homewood, Ill.: Richard D. Irwin, Inc., 1954.

Wilkinson, J. H. *The Algebraic Eigenvalue Problem.* Oxford, England: Clarendon Press, 1965.

Wonnacott, R. J. *Canadian-American Dependency, An Interindustry Analysis of Production and Prices.* Amsterdam: North-Holland Publishing Co., 1961.

Yamada, I. *Theory and Application of Interindustry Analysis.* Tokyo: Kinokuniya Bookstore Co., Ltd., 1961. (Hitotsubashi University, The Institute of Economic Research, Economic Research Series No. 4).

C. PERIODICAL LITERATURE

Adams, A. A., and Stewart, I. G. "Input-Output Analysis: An Application." *Economic Journal,* LXVI (September, 1956), 442-454.

Amato, Vittorio. "Rappresentazione matriciale del 'multiplicatore' dinamico," *L'industria rivista di economia politica,* No. 3 (1961), 367-379.

Ara, Kenjiro. "The Aggregation Problem in Input-Output Analysis," *Econometrica,* XXVII, 2 (April, 1959), 257-262.

Artle, Roland. "External Trade, Industrial Structure, Employment Mix, and the Distribution of Incomes: A Simple Model of Planning and Growth," *The Swedish Economic Journal* (Ekonomisk Tidskrift), LXVII, 1 (March 1965), 1-23.

————. "On Some Methods and Problems in the Study of Metropolitan Economies," *Regional Science Association: Papers,* VIII (1962), European Congress, The Hague, 1961, 71-92.

Aujac, H. "La hiérarchie des industries dans un tableau des échanges interindustriels," *Revue économique,* XI, 2(March, 1960), 169-238.

Bacharach, M. "Estimating Nonnegative Matrices from Marginal Data," *International Economic Review,* VI, 3 (September, 1965), 294-310.

Barankin, E. W. "Bounds for the Characteristic Roots of a Matrix," *Bulletin of the American Mathematical Society,* LI (1945), 767-770.

Bear, D. V. T. "The Matrix Multiplier and Distributed Lags," *Econometrica,* XXXI, 3 (July, 1963), 514-529.

Bear, D. V. T., Jorgenson, D., and Wagner, H. M. "Elementary Proofs of Propositions on the Leontief-Minkowski Matrices," *Metroeconomica,* XIV, 1-2-3 (April-August-December, 1962), 59-64.

Berger, W. J., and Saibel, Edward. "Power Series Inversion of the Leontief Matrix," *Econometrica,* XXV, 1 (January, 1957), 154-165.

Bernadet, R. Jouandet. "Le Tableau Interindustriel Regional," *Revue économique,* XVIII, 3 (May, 1967), 447-479.

Box, G. E. P., and Hunter, J. S. "A Confidence Region for the Solution of a Set of Simultaneous Equations with an Application to Experimental Design," *Biometrika,* XLI (1954), 190-199.

Brauer, Alfred. "Limits for the Characteristic Roots of a Matrix," *Duke Mathematical Journal,* XIII (1946), 387-395.

————. "A Method for the Computation of the Greatest Root of a Nonnegative Matrix," *Journal of the Society for Industrial and Applied Mathematic Series B, Numerical Analysis* (SIAM J. NUMER, ANAL.), III, 4 (1966), 564-569.

————. "The Theorems of Ledermann and Ostrowski on Positive Matrices," *Duke Mathematical Journal,* XXIV (1957), 265-274.

Bray, Hubert E. "Rates of Exchange," *American Mathematical Monthly,* XXIX (1922), 365-371.

Brualdi, Richard A. "On the Permanent and Maximal Characteristic Root of a Nonnegative Matrix," *Proceedings of the American Mathematical Society,* XVII, 6 (December, 1966), 1413-1416.

Bruno, Michael. "Optimal Patterns of Trade and Development," *The Review of Economics and Statistics,* XLIX, 4 (1967), 545-554.

Cameron, Burgess. "The Production Function in Leontief Models." *The Review of Economic Studies,* XX, 1 (1952), 62-69.

Carlson, David. "A Note on M-Matrix Equations," *Journal of the Society for Industrial and Applied Mathematics,* XI, 4 (December, 1963), 1027-1033.

Carter, Anne P. "Changes in the Structure of the American Economy, 1947 to 1958 and 1962," *The Review of Economics and Statistics,* XLIX, 2 (May, 1967), 209-224.

Chakravarty, Sukhamoy. "The Matrix Extension of the Multiplier," *The Indian Economic Review,* III, 3 (February, 1957), 41-52.

Chipman, J. S. "The Multi-Sector Multiplier," *Econometrica,* XVIII, (October, 1950), 355-374.

Chipman, John S. "Professor Goodwin's Matrix Multiplier," *The Economic Journal,* LX (December, 1950), 753-770.

Cornfield, Jerome, Evans, W. Duane, and Hoffenberg, Marvin. "Full Employment Pattern, 1950." *Monthly Labor Review,* LXIV, (February-March, 1947), 163-190; 420-432.

Crabtree, Douglas E. "Applications of M-Matrices to Non-Negative Matrices," *Duke Mathematical Journal,* XXXIII, (1966), 197-207.

Crabtree, Douglas E. "On the Characteristic Roots of Matrices," *Proceedings of the American Mathematical Society,* XVI, (1965), 1410-1413.

Dappe and Delange *(sic).* "Traitement des Commerces et des Transports dans les Tableaux d' échanges interindustriels," *Etudes de Comptabilité nationale,* 3 (1962), 6-33.

Debreu, Gerard, and Herstein, I. N. "Nonnegative Square Matrices," *Econometrica,* XXI, 4 (October, 1953), 597-607.

Dorfman, Robert. "The Nature and Significance of Input-Output," *The Review of Economics and Statistics,* XXXIV, 2 (May, 1954), 121-133.

Edmonston, J. Harvey. "A Treatment of Multiple-Process Industries," *The Quarterly Journal of Economics,* LXVI, 4 (November, 1952), 557-571.

Egervary, E. "On a Lemma of Stieltjes on Matrices," *Acta Scient. Math. Szeged,* XV, (1953-1954), 99-103.

Etherington, I. M. H. "On Errors in Determinants," *Proceedings of the Edinburgh Mathematical Society,* III (1932), 107-117.

Evans, W. Duane, and Hoffenberg, Marvin. "The Interindustry Relations Study for 1947," *The Review of Economics and Statistics,* XXXIV, 2 (May, 1952), 97-142.

Fan, Ky. "Notes on M-Matrices," *Quarterly Journal of Mathematics,* XI (1960), 43-49.

————. "Topological Proofs for Certain Theorems on Matrices with Non-negative Elements," *Monatshefte für Mathematik,* LXII, (1958), 219-242.

Fan, Ky., and Householder, A. S. "A Note Concerning Positive Matrices and M-Matrices," *Monatshefte für Mathematik,* LXIII (1959), 265-270.

Fei, John C. H. "A Fundamental Theorem for the Aggregation Problem of Input-Output Analysis," *Econometrica,* XXIV, 4 (October, 1956), 400-412.

————. "A Preliminary Input-Output Table for Large-Scale Industries in Pakistan," *The Pakistan Development Review,* II, 1 (Spring, 1962), 47-83.

Fiedler, M., and Pták, V. "On Matrices with Non-positive Off-Diagonal Elements and Positive Principal Minors," *Chekhoslovatskii Matematicheskii Zhurnal,* XII, (1962), 382-400.

Fisher, Franklin M. "An Alternate Proof and Extension of Solow's Theorem on Nonnegative Square Matrices," *Econometrica,* XXX, 2 (April, 1962), 349-350.

Fisher, Walter D. "Criteria for Aggregation in Input-Output Analysis," *The Review of Economics and Statistics,* XL, 3 (August, 1958), 250-260.

Freutel, Guy. "The Eighth District Balance of Trade," *Monthly Review,* Federal Reserve Bank of St. Louis, (June 1952), 69-78.

Frobenius, G. "Über Matrizen aus nicht negativen Elementen," *Sitzungsberichte der koniglich preussischen Akademie der Wissenschaften zu Berlin,* No. 26-27 (1912), 456-477.

————. "Über Matrizen aus positiven Elementen," *Sitzungsberichte der koniglich preussischen Akademie der Wissenschaften zu Berlin,* No. 26 (1908), 471-476.

Frumkin, Norman. "Construction Activity in the 1958 Input-Output Study," *Survey of Current Business,* XLV, 5 (May, 1965), 13-24.

Gautschi, W. "The Asymptotic Behavior of Powers of Matrices I and II," *Duke Mathematical Journal,* XX, (1953), 172-140 and 375-379.

Gerschgorin, S. "Über die Abgrenzung der Eigenwerte einer Matrix," *Bulletin de l'Academie des Sciences de l'URSS, Classe Mathematique,* 7th serie (1931), 749-754.

Ghosh, A. "Input-Output Analysis with Substantially Independent Groups of Industries," *Econometrica,* XXVIII, 1 (1960), 88-96.

————. "Interregional Models and the Problem of Substitution," *The Econometric Annual of the Indian Economic Journal,* XIV, 2 (1967), 212-232.

Ginsburgh, V. "Critéres théoriques et pratiques de l'agrégation dans input-output et validité de l'agrégation adoptée dans le modéle de croissance de Bruxelles," *Cahiers Economiques de Bruxelles, 25,* 1er trimestre (1965), 106-123.

Goheen, H. E. "On a Lemma of Stieltjes on Matrices," *American Mathematical Monthly,* LVI, (1949), 328-329.

Goldman, Morris R., Marimont, Martin L., and Vaccara, Beatrice N. "The Interindustry Structure of the United States, A Report on the 1958 Input-Output Study," *Survey of Current Business,* XLIV, 11 (November, 1964), 10-20.

Goodwin, R. M. "Does the Matrix Multiplier Oscillate," *The Economic Journal,* LX (December, 1950), 764-770.

————. "The Multiplier as Matrix," *The Economic Journal,* LIX, (December, 1949), 537-555.

Hagger, A. "Matrix-Multiplier Analysis: An Exposition," *Yorkshire Bulletin of Economic and Social Research,* X, 1 (June, 1958), 19-34.

Hansen, W. Lee, and Tiebout, Charles M. "An Intersectoral Flows Analysis of the California Economy," *The Review of Economics and Statistics,* XLV, 4 (November, 1963), 409-418.

Hatanaka, M. "Note on Consolidation within a Leontief System," *Econometrica,* XX, 2 (April, 1952), 301-303.

Hawkins, David, and Simon, Herbert A. "Note: Some Conditions of Macroeconomic Stability," *Econometrica,* XVII, 3 and 4 (July-October, 1949), 245-248.

Helmstadter, Ernst. "Die geordnete Input-Output struktur," *Jahrbücker fürNationalökonomie und Statistik,* CLXXIV, 4-5-6 (1962), 322-361.

Herstein, I. N. "A Note on Primitive Matrices," *American Mathematical Monthly,* LXI (1954), 18-20.

Hirsch, Werner Z. "An Application of Area Input-Output Analysis," *The Regional Science Association: Papers and Proceedings,* V, (1959), 79-92.

————. "Input-Output Techniques for Urban Government Decisions," *American Economic Review,* LVIII, 2 (May, 1968), 162-170.

————. "Interindustry Relations of a Metropolitan Area," *The Review of Economics and Statistics,* XLI, 4 (Nov., 1959), 360-369.

Hoch, Irving. "A Comparison of Alternative Inter-Industry Forecasts for the Chicago Region," *The Regional Science Association: Papers and Proceedings,* V, (1959), 217-235.

Hochwald, Werner. "Dependence of Local Economies Upon Foreign Trade: A Local Impact Study," *The Regional Science Association: Papers and Proceedings,* IV (1958), 259-271.

Holladay, J. C., and Varga, R. S. "On Powers of Non-negative Matrices," *Proceedings of the American Mathematical Society,* IX (1958), 631-634.

Householder, A. S. "On Matrices with Non-Negative Elements," *Monatshafte für Mathematik,* LXII (1958), 238-242.

Hurwicz, Leonid. "Input-Output Analysis and Economic Structure: A Review Article," *The American Economic Review,* XLV, 4 (September, 1955), 626-636.

Isard, Walter. "Interregional and Regional Input-Output Analysis: A Model of a Space Economy," *The Review of Economics and Statistics,* XXXIII, (November,1951), 318-328.

————. "Regional Commodity Balances and Interregional Commodity Flows," *The American Economic Review,* XLIII, 2 (May, 1953), 167-180.

Isard, Walter, and Kuenne, Robert E. "The Impact of Steel Upon the Greater New York-Philadelphia Industrial Region," *The Review of Economics and Statistics,* XXXV, (November, 1953), 289-301.

Isard, Walter, and Kavesh, Robert. "Economic Structural Interrelations of Metropolitan Regions," *American Journal of Sociology,* LX, 2 (September, 1954), 152-162.

Iyemoto, Hidetaro. "Note on Input-Output Analysis: Differences between the Repurcussion Effects in Physical Terms and in Value Terms." *Econometrica,* XXVIII, 3 (July, 1960), 699-700.

Kossov, V. V. "Regional Input-Output Analysis in the U.S.S.R.," *Regional Science Association: Papers,* XIV, Ghent Congress (1964) 175-181.

Kotelyanskii, D. M. "Some Properties of Matrices with Positive Elements," *Matematicheskii Sbornik,* XXXI, 73 (1952), 497-506.

Kundu, A. "Interindustry Table for the Economy of British Guiana, 1959 and National Accounts, 1957-1960," Supplement to XII, 1. *Social and Economic Studies,* March, 1963.

Ledermann, Walter. "Bounds for the Greatest Latent Roots of a Positive Matrix," *The Journal of the London Mathematical Society,* XXV, (1950), 265-268.

Leontief, Wassily. "Quantitative Input-Output Relations in the Economic System of the United States," *The Review of Economics and Statistics,* XVIII, 3 (August, 1936), 105-125.

Leontief, Wassily, *et al.*, "The Economic Impact – Industrial and Regional – of an Arms Cut," *The Review of Economics and Statistics*, (XLVII, (August, 1965), 217-242.

Malinvaud, Edmond. "L'agrégation dans les modèles économiques," *Cahiers du Seminaire d'Econometrie,* 4 (Paris, 1956), 69-146.

Matuszewski, T. I. "On Some Input-Output Computations," *Journal of the Royal Statistical Society,* Series A (General), CXXIII, Part 2 (1960), 195-199.

Matuszewski, T. I., Pitts, P. R., and Sawyer, John A. "Linear Programming Estimates of Changes in Input Coefficients," *The Canadian Journal of Economics and Political Science,* XXX, 2 (May, 1964), 203-210.

Metzler, L. A. "A Multiple-Region Theory of Income and Trade," *Econometrica,* XVIII, (October, 1950), 329-354.

————. "Stability of Multiple Markets: The Hicks Conditions," *Econometrica,* XIII, (October, 1945), 277-292.

————. "Taxes and Subsidies in Leontief's Input-Output Model," *The Quarterly Journal of Economics,* LXV (August, 1951), 433-438.

Meyer, John. "Regional Economics: A Summary," *The American Economic Review,* LIII, (March, 1963), 19-54.

McCarthy, John. "Aggregation in the Open Leontief Model," *Econometrica,* XXV, 4 (October, 1957), 602.

McCarthy, Michael D. "On the Aggregation of the 1958 Direct Requirements Input-Output Table," *The Review of Economics and Statistics,* XLIX, 4 (November, 1967), 600-602.

McKenzie, Lionel. "An Elementary Analysis in the Leontief System," *Econometrica,* XXV, 3 (July, 1957), 456-462.

McManus, M. "General Consistent Aggregation in Leontief Models," *Yorkshire Bulletin,* VIII, (June, 1956), 28-48.

Miller, Ronald E. "The Impact of the Aluminum Industry on the Pacific Northwest: A Regional Input-Output Analysis," *The Review of Economics and Statistics,* XXXIX (May, 1957), 200-209.

Minkowski, H. "Zur Theorie der Einheiten in den algebraischen Zahlkorpern," *Nachrichten von der Koniglichen Gesellschaft der Wissenschaften zu Gottingen, Mathematisch-physikalische Klasse,* (1900), 90-93.

Miyazawa, K. "Internal and External Matrix Multipliers in the Input-Output Model," *Hitotsubashi Journal of Economics,* Kunitachi, Tokyo, VII, 1 (June, 1966), 38-55.

Moore, Frederick T. "Regional Economic Reaction Paths," *American Economic Review,* XLV, (May, 1955), 133-155.

Moore, F. T., and Petersen, James W. "Regional Analysis: An Interindustry Model of Utah," *The Review of Economics and Statistics,* XXXVII, (November, 1955), 368-380.

Morishima, M. "Generalizations of the Frobenius-Wielandt Theorems for Non-negative Square Matrices," *Journal of the London Mathematical Society,* XXXVI, (1961), 211-220.

————. "On the Laws of Change of the Price System in an Economy Which Contains Complementary Commodities," *Osaka Economic Papers,* I (1952), 101-113.

Moses, Leon. "The Stability of Interregional Trading Patterns and Input-Output Analysis," *The American Economic Review,* XLV, (December, 1955), 803-832.

————. "A General Equilibrium Model of Production, Interregional Trade and Location of Industry," *The Review of Economics and Statistics,* XLII, (November, 1960), 373-397.

Ostrowski, A. M. "Bounds for the Greatest Latent Root of a Positive Matrix," *The Journal of the London Mathematical Society,* XXVII (1952), 253-256.

————. "Determinanten mit Überwiegender Hauptdiagonale and die absolute Konvergenz von linearen Iterationsprozessen," *Commentarii Mathematici Helvetici,* XXX, (1956), 175-210.

————. "On Positive Matrices," *Mathematische Annalen,* CL, (1963), 276-284.

————. "Über die Determinanten mit Überwiegender Hauptdiagonale," *Commentarii Mathematici Helvetici,* X, (1937), 69-96.

Ostrowski, A. M., and Schneider, Hans. "Bounds for the Maximal Characteristic Root of a Non-negative Irreducible Matrix," *Duke Mathematical Journal,* XXVII, (1960), 547-553.

Paelinck, J., and Waelbroeck, J. "Etude empirique sur l'èvolution de coefficients 'input-output'," *Economie Appliquée,* XIV, (January-March, 1963), 81-111.

Perron, Oskar. "Zur Theorie der Matrizes," *Mathematische Annalen,* LXIV, (1907), 248-263.

Peterson, Richard S., and Tiebout, Charles M. "Measuring the Impact of Regional Defense - Space Expenditures," *The Review of Economics and Statistics,* XLVI, 4 (November, 1964), 421-428.

Phillips, Almarin. "The Tableau Economique as a Simple Leontief Model," *The Quarterly Journal of Economics,* LXIX, 1 (February, 1955), 137-144 .

————. "The Variation of Technical Coefficients in the Antifriction Bearing Industry." *Econometrica,* XXIII, (October, 1955), 432-441.

Ponsard, Claude. "Essai D'Interpretation Topologique Des Systemes Interregionaux," *Revue économique,* XVIII, 3 (Mai, 1967), 353-373.

Pták, V. "On a Combinatorial Theorem and its Application to Non-negative Matrices," *Chekhoslovatskii Matematicheskii Zhurnal,* VIII, (1958), 487-495.

Pták, V., and Sedláček, J. "On the Index of Imprimitivity of Nonnegative Matrices," *Chekhoslovatskii Matematicheskii Zhurnal,* VIII, (1958), 496-501.

Quandt, R. E. "On the Solution of Probabilistic Leontief Systems," *Naval Research Logistics Quarterly,* VI, (1959), 295-305.

————. "Probabilistic Errors in the Leontief System," *Naval Research Logistics Quarterly,* V, (1958), 155-170.

Rey, G., and Tilanus, C. B. "Input-Output Forecasts for the Netherlands, 1949-1958." *Econometrica,* XXXI, (1963), 454-463.

Rohrback, H. "Bemerkunger Zueinem Determinantensatz von Minkowsky," *Jahresbericht den Deutschen Mathematiker Vereinigung,* XL, (1931), 49-53.

Romanovsky, V. "Recherches sur les chains de Markoff," *Acta Mathematica,* LXVI, (1936), 147-251.

Rosenblatt, David. "On Linear Models and the Graphs of Minkowski-Leontief Matrices," *Econometrica,* XXV, 2 (April, 1957), 325-338.

Samelson, H. "On the Perron-Frobenius Theorem," *Michigan Mathematical Journal,* IV, (1957), 57-59.

Sawyer, John A. "The Measurement of Inter-Industry Relationships in Canada." *The Canadian Journal of Economics and Political Science,* XXI, 4 (November, 1955), 480-497.

Seton, F. "Static Multipliers and Social Accounting Systems," *Oxford Economic Papers,* XVI, 3 (November, 1964), 297-320 .

Shen, T. Y. "An Input-Output Table with Regional Weights," *The Regional Science Association: Papers and Proceedings,* VI, (1960), 113-119.

Simon, Nancy W. "Personal Consumption Expenditures in the 1958 Input-Output Study," *Survey of Current Business,* XLIV, 10 (October, 1965), 7-20,28.

Simpson, David, and Tsukui, Jinkichi. "The Fundamental Structure of Input-Output Tables: An International Comparison," *The Review of Economics and Statistics,* XLVIII, 4 (November, 1965), 434-443.

Smith, Vernon L. "Engineering Data and Statistical Techniques in the Analysis of Production and Technological Change: Fuel Requirements in the Trucking Industry." *Econometrica,* XXV, (April, 1957), 281-301.

Solow, Robert M. "A Note on Dynamic Multipliers," *Econometrica,* XIX, 3 (July, 1951), 306-316.

————. "On the Structure of Linear Models," *Econometrica,* XX, 1 (January, 1952), 29-46.

Specht, W. "Zur Theorie der Algebraischen Gleichungen," *Jahresbericht der Deutschen Mathematiker Vereinigung,* XLVIII, (1938), 142-145.

Stewart, I. G. "Input-Output Table for the United Kingdom." *The Times Review of Industry, London and Cambridge Economic Bulletin,* New Series., No. 28, (1958), 29-77.

Taussky, Olga. "Bounds for Characteristic Roots of Matrices," *Duke Mathematical Journal,* XV, (1948), 1043-1044.

————. "Bounds for Characteristic Roots of Matrices II," *Journal of Research of the National Bureau of Standards,* XLVI, 2, Research Paper 2184, (February, 1951), 124-125.

Theil, H. "Linear Aggregation in Input-Output Analysis," *Econometrica,* XXV, 1 (January, 1957), 111-122.

Theil, Henri, and Uribe, Pedro. "The Information Approach to the Aggregation of Input-Output Tables," *The Review of Economics and Statistics,* XLIV, 4 (November, 1967), 451-461.

Tiebout, Charles M. "Regional and Interregional Input-Output Models: An Appraisal," *The Southern Economic Journal,* XXIV, (October, 1957), 140-147.

Ullman, J. L. "On a Theorem of Frobenius," *Michigan Mathematical Journal,* I, (1952), 189-193.

U. S. Department of Commerce, Office of Business Economics, National Economics Division Staff. "The Transactions Table of the 1958 Input-Output Study, Revised Direct and Total Requirements Data," *Survey of Current Business,* XLIX, 9 (September, 1965), 33-49, 56.

U. S. Department of Commerce, Office of Business Economics, National Economics Division Staff. "Additional Industry Detail for the 1958 Input-Output Study," *Survey of Current Business,* XLVI, 4 (April, 1966), 14-17.

Waelbroeck, J. "Meccano: or a do-it-yourself approach to long-term forecasting [sic]," *Cahiers Economiques de Bruxelles,* (1966), 203-242.

Wielandt, H. "Unzerlegbare, nicht negative Matrizen," *Mathematische Zeitschrift,* LII, (March, 1950), 642-648.

Wong, Y. K. "On Nonnegative-valued Matrices," *Proceedings of the National Academy of Sciences,* XL, (1954), 121-124.

————. "Quasi-Inverses Associated Minkowski-Leontief Matrices," *Econometrica,* XXII, 3 (July, 1954), 351-352.

Wong, Y. K. ,and Morgenstern, Oskar. "A Study of Linear Economic Systems," *Weltwirtschaftliches Archiv,* Band 79 (1957 II), 222-239.

Yamada, Isamu. "Regional Input-Output Analysis – An Interpretation of Chenery's Model," *Hitotsubashi Journal of Economics,* III, 2 (June, 1963), 1-5.

Yamamoto, Tetsuro. "On the Extreme Values of the Roots of Matrices," *Journal of the Mathematical Society of Japan,* XIX, 2 (April, 1967), 173-178.

D. MISCELLANEOUS REPORTS, PAMPHLETS, PAPERS AND UNPUBLISHED PH.D. THESES

Allen, Robert S., and Watson, Donald A. *The Structure of the Oregon Economy: An Input/ Output Study.* Eugene, Oregon: The University of Oregon Press, 1965.

Almon, Clopper, Jr. "New Developments in Input-Output Forecasting." Paper presented at meetings of the Institute for Management Science in Boston, April 7, 1967.

Arrow, Selma. *Comparisons of Input-Output and Alternative Projections, 1929-39.* Santa Monica, California: The RAND Corporation, 14 April 1951.

Berman, B. R., Chinitz, B., and Hoover, E. M. *Projection of a Metropolis: Technical Supplement to the New York Metropolitan Region Study.* Cambridge, Mass.: Harvard University Press, 1960.

Carden, John G. D., and Whittington, F. B., Jr. *Studies in the Economic Structure of the State of Mississippi.* Jackson, Miss.: Mississippi Industrial and Technological Research Commission, 1964. Two Volumes.

Carter, Harold O., and Heady, Earl O. *An Input-Output Analysis Emphasizing Regional and Commodity Sectors of Agriculture.* Ames, Iowa: Iowa State University, Agricultural and Home Economics Experiment Station, (September, 1959), Research Bulletin 469.

Chapman, John Howard. "The Effects of Trade Pattern Changes on Regional Input-Output Analysis." Unpublished Ph. D. Thesis, University of Colorado, 1965.

Chapman, J. H., Jr., and Schellhammer, K. L. *The Structure of the West Virginia Economy 1965.* A Preliminary Report. Morgantown, West Va.: Regional Research Report, West Virginia University, November, 1967 (mimeographed).

Cumberland, J. H. "Examples of Variations in the Behavior of Critical Material Input Coefficients," Interindustry Item No. 17, Washington, D. C.: U. S. Bureau of Mines, 1952.

Davis, Harrell Craig. "A Multiregional Input-Output Model of the Western States Emphasizing Heavy Water-Using Sectors." Unpublished Ph. D. Thesis, University of California, Berkeley, Calif., 1967.

Dhar, R. *An Input-Output Table for West Bengal and the Calcutta Metropolitan District: First and Second Interim Reports.* New York: Calcutta Research Studies, Occasional Report, Institute of Public Administration, N. Y., 1962 (Mimeographed).

Eckaus, Richard S., and Parikh, K. S. *Planning for Growth: Multi-sectoral, Intertemporal Models Applied to India.* Cambridge, Mass.: Massachusetts Institute of Technology, Center for International Studies, 1966 (Mimeographed).

Goldstein, Harold T. *Historical Comparability of Census of Manufactures Industries, 1929-1958,* U. S. Bureau of the Census Working Paper No. 9, Washington, D. C.: U. S. Government Printing Office, 1959.

Helzner, M. L. "A Study of Coefficient Variation of Selected Inputs into the Steel Industry." Interindustry Item No. 48 (Washington, D. C.: U. S. Bureau of Mines, 1954).

Isard, W., Langsford, T. W., Jr., and Romanoff, E. *Philadelphia Region Input-Output Study.* Philadelphia, Pennsylvania: Department of Regional Science, Wharton School, University of Pennsylvania, and Regional Science Research Institute, 2 Vols., 1966.

Kirksey, C. D. *An Inter-Industry Study of the Sabine-Neches Area of Texas.* Austin, Texas: Bureau of Business Research, University of Texas, 1959.

Koenig, Lou P., and Ritz, Philip M. *Secondary Product Adjustment with Redistribution (SPAR).* National Planning Association, Economic Programming Center, NREC Technical Report No. 67, A Report to the U. S. Army Engineer Mathematical Computation Agency, Corps of Engineers, Pursuant to Contract No. DA-18-020-ENG-3628, Task I-A (April, 1967), 117 pp.

Lee, Tong Hum, Moore, John R., and Lewis, David P. *A Report on the Tennessee Interindustry Study.* Knoxville, Tennessee: The University of Tennessee, College of Business Administration, Center for Business and Economic Research, December 1967.

Lund, R. E. *An Analysis of a Local Economy in a Period of Rapid Transition, Southwestern Wyoming.* Laramie, Wyo.: University of Wyoming, College of Commerce and Industry, Division of Business and Economic Research, June, 1962.

Marshall, A. W. *Comments on H. J. Barnett's "Specific Industry Output Projections."* Santa Monica, California: The RAND Corporation, 1 October 1951.

Martin, William E., and Carter, Harold O. *A California Interindustry Analysis Emphasizing Agriculture.* University of California, California Agriculture Experiment Station, Giannini Foundation of Agricultural Economics, February 1962, Giannini Foundation Research Report No. 250.

McKie, James W. *Industry Classification and Sector Measures of Industrial Production,* U. S. Bureau of the Census Working Paper No. 20. Washington, D. C.: U. S. Government Printing Office, 1965.

Moody, Harold T. "Theory and Application of Regional Income and Product Accounts," Unpublished Ph.D. Thesis, University of California–Los Angeles, Department of Business Administration, 1964.

The Netherlands Central Bureau of Statistics, *Input-Output Tables for the Netherlands,* Statistical Studies, No. 16. Zeist, Netherlands: Uitgeversmaatschappij W. de Haan N. V., July, 1963.

Polenske, Karen and Tsukui, Jinkichi. "Consistent Estimate for the Leontief Gravity Model of Interregional Trade." Harvard Economic Research Project No. 1583. January 1965.

Riefler, Roger Frank. "Interregional Input-Output: Washington and California." Unpublished Ph. D. Thesis, University of Washington, 1966.

Ritz, Philip M. (ed.). *Input-Output Analysis, Technical Supplement.* Conference on Research in Income and Wealth. New York: National Bureau of Economic Research, Inc., 1954.

Ritz, Philip M., and Rudney, Gabriel G. *General Explanations of the 200 Sector Tables: The 1947 Interindustry Relations Study,* U. S. Department of Commerce, Bureau of Labor Statistics, BLS Report No. 33, June, 1953.

————. *The 1947 Interindustry Relations Study, Industry Reports, General Explanations.* U. S. Department of Labor, Bureau of Labor Statistics, Division of Interindustry Economics, BLS Report No. 9, March, 1953.

Rubin, Rose Mohr. "Aggregation Criteria in Input-Output Analysis." Unpublished Ph. D. Thesis, Kansas State University, 1968.

Shishido, S. "Recent Input-Output Studies in Japan—Particularly in Government Agencies," Stanford Project for Quantitative Research in Economic Development at Stanford University, Memorandum Number C-6. Stanford University: October, 1957, 10-17.

State of Hawaii, Department of Planning and Economic Development. *Patterns of Economic Growth: The State Economic Model State of Hawaii General Plan Revision Program,* Part 3 (Honolulu: 1967).

Strout, Alan M. "A Flexible Input-Output Convention for Secondary Product Transactions." Cambridge, Mass.: Harvard University, Harvard Economic Research Project, 14 January 1963 (Mimeographed).

————. "Dissaggregation of an Industry Production Function When it is Desired to Treat Individual Industry Joint Products in Separate Input-Output Table Rows." Cambridge, Mass.: Harvard University, Harvard Economic Research Project, 24 October 1962 (Mimeographed).

United Nations. *Problems of Input-Output Tables and Analysis,* New York: United Nations, 1966.

U. S. Department of Commerce, Office of Business Economics, National Income Division. *The 1958 Interindustry Relations Study.* Unpublished Preliminary Report, November, 1964.

U. S. Department of Commerce, Office of Business Economics. *Input-Output Transactions: 1961.* Staff Working Paper in Economics and Statistics, No. 16, July, 1968.

U. S. Department of Labor, Bureau of Labor Statistics. *Projections 1970, Interindustry Relationships,* Potential Demand, Employment, BLS Bulletin No. 1536. Washington, D. C.: U. S. Government Printing Office, 1966.

Wang,Kung-Lee, and Kokat, Robert G. *The Interindustry Structure of the U. S. Mining Industries, 1958.* U. S. Department of the Interior, Bureau of Mines, Information Circular 8338. Washington, D. C.: U. S. Government Printing Office, 1967.

BIOGRAPHICAL NOTE

Name: Kirkor Bozdogan

I. PERSONAL

Date of Birth:	March 4, 1938
Birthplace:	Felahiye, Kayseri, Turkey
Marital Status:	Married
Number of Children:	None
Current Address:	141 Magazine Street, Cambridge, Mass. 02139

II. EDUCATIONAL BACKGROUND

B.S., 1961,	University of Wisconsin, Madison, Wisconsin
M.C.P., 1963,	Harvard University, Cambridge, Mass.

III. TEACHING EXPERIENCE

Teaching Assistant, Massachusetts Institute of Technology, Department of City and Regional Planning.

Spring Term, 1965-1966:	Course 4.74 – Methods of Research in City Planning (Prof. A. Fleisher)
Spring Term, 1966-1967:	Course 11.90 – Methods of Research in City Planning (Prof. A. Fleisher and Dr. I.S. Lowry)

IV. PROFESSIONAL EXPERIENCE

Professional staff member in the Corporate and Public Management Division of Arthur D. Little, Inc., since 1963.

V. PROFESSIONAL CONTRIBUTIONS, INTERESTS, AND CURRENT ACTIVITIES

- Contributions to numerous studies at Arthur D. Little, Inc., in the following areas: national and regional economic forecasting, economic impact analysis, industrial development planning, educational and manpower planning.
- Interests in the following areas: mathematical/econometric model building for long- and short-term multisectoral economic forecasting and public policy simulation; quantitative research of urban and regional problems, with particular interest in economic forecasting, economic impact analysis, housing market simulation, manpower planning, public policy issues relating to poverty, and ecological-economic development studies.
- Current activities in the following areas: development of a short-term multisectoral economic forecasting model for the U.S. economy, the economics of weather forecasting, resource allocation for nautical mapping, charting and oceanographic research, analysis of public policy issues relating to ecological-economic development problems, urban and regional economic modeling.